# PHY 1100 General Physics 1/PHY 1300 Advanced General Physics 1

## Part 1: Math for science

**"Good – he did not have enough imagination to become a mathematician." David Hilbert (1862 – 1943) upon being told that a student in his class had dropped mathematics in order to become a poet.**

Using Math: Many students promptly forget their math classes, as soon as they finish them. That's not OK in science. You may have thought during your math class, "Who uses algebra in real life?" Scientists do! In Science, you are literally required to remember and use all of the math that you have ever taken, from kindergarten to calculus. Some science professors are willing to review math, but most are not.

Using a SCIENTIFIC CALCULATOR: You are required to buy and learn how to use a scientific calculator. No one does long calculations by hand anymore in science/ engineering, unless they're showing off. Some scientists, especially Physicists, do show off, but we don't need you to show off. You do not need a $150 calculator, unless you intend to become a professional scientist or engineer. The Texas Instruments TI-30Xa is less than $20; the TI-36X is less than $25. Your smartphone has a built-in calculator app that you may use in an emergency, but you may not use it for tests.

ABBREVIATIONS: Scientists are very lazy. They refuse to write anything out completely; they abbreviate everything. Every letter of the alphabet, upper and lower case, stands for something. Students must continually practice using them - for measurements, for units or for technical terms - so that they become second nature. Some students find flash cards helpful to remember factoids like abbreviations.

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### the metric system

Scientists measure EVERYTHING, and they always use the METRIC SYSTEM (the Système International). Unfortunately, the United States is almost literally the last country on Earth that still uses British (or Imperial) units for ordinary purposes. (Even the British don't use British units anymore.)

Students must be comfortable with the metric system. This means being familiar with real things in metric units, such as - a paper clip weighs about 1 gram, a high ceiling is about 3 meters, a 40°C day is extremely hot and uncomfortable, a 50 square meter apartment is good for New York, driving 105 kilometers per hour is the highway speed limit, or that a gallon of milk weighs 4 kilograms.

This also means knowledge of the metric prefixes and how they relate - by converting - related units, like meter, kilometer and centimeter.

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## scientific notation

When scientists measures everything, they often run into extremely large or exceptionally small things. The measurement will end up with many digits, especially zeroes, which will be hard to keep track of.

Scientists will abbreviate very long numbers, by counting decimal places and rewriting in scientific notation.

Scientific notation always looks like this:

$$x \times 10^y$$

For example: 149,600,000,000 $m \;=\; 1.496 \times 10^{11}\ m$

Scientific calculators can understand and use scientific notation. Numbers with more digits than can fit on the screen - usually ten - must be in scientific notation. Be careful, scientific notation is NOT A MULTIPLICATION; it is an abbreviation. DO NOT press the multiplication button, or enter ten, when you use scientific notation on your calculator. Your calculator will try to multiply, which is not what you're trying to do. Use the scientific notation button; usually labeled "EE", "EXP", or "$\times 10^y$". Your calculator's screen may not show the $\times 10$ either. There may only be an empty space or an "E" between the mantissa and exponent.

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## significant figures

Science is not math. In science, numbers do not count; they measure, and measurements have a precision limited by the quality of the measuring equipment.

Significant figures is rounding off after a calculation to the proper precision. Calculators often give ten decimal places solutions. In science, most of the digits are irrelevant, because the original measurements don't have that many places. In science, the answer cannot be more precise than the information.

The rounding procedure is to ROUND UP if the next digit is GREATER THAN FIVE (5); ROUND DOWN if the next digit is LESS THAN FIVE (5); and ROUND TO AN EVEN DIGIT if the next digit is EXACTLY FIVE (5 or 500…). Note: this scientific rounding procedure is different than that taught in Math class.

Remember: calculators DO NOT know how to use significant figures.

Remember also the order of operations: if there are no parentheses powers (exponents) are calculated first, multiplication and division next, and addition and subtraction is last.

On the responsibility of computer/calculator use in science and engineering: “If the computer screws up - no, it didn’t. You screwed up, because you weren’t paying attention.” A recent example is the Airbus A380, the largest airliner in the world - 550 passengers, 9000 miles, 1,200,000 lbs - first ticket 2007. Airbus is a European consortium - a company made up of cooperating companies. Designing a modern airliner is so complex - requiring many thousands of engineers - that no single European country is big enough to design one. Therefore, different parts of the A380 were designed by different members of the consortium: the wings were designed in England, the tail in Italy, the landing gear in Spain, and the front half of the fuselage in France and the back half in Germany. In modern engineering, prototyping - building improving iterations of real machines for testing - is minimized to save money. The A380 had no prototype - the airliner was completely designed and tested on computers. It turned out the French were using version 4 of their computer engineering program, while the Germans were using version 5. The software company changed how the program rounded off calculations between the two versions - this was clearly stated in the documentation. If everyone had used the same version, everything would have been OK, but when the first A380 was built, it turned out that front and back halves of the fuselage DIDN’T FIT TOGETHER!!! It took Airbus a whole year and $1 billion to redo all the work; their CEO was fired! Ironically, the A380 was discontinued in 2019, due to poor sales - only 274 units sold. It turned out that A380 was too big, and Airbus won’t make a profit on it. (PHY 1300 only)

# Part 2: Linear motion

**"The whole history of science has been the gradual realization that events do not happen in an arbitrary manner, but that they reflect a certain underlying order." Stephen Hawking (1944 – 2018)**

Physics is the study of MATTER and ENERGY and their interactions. What is matter and energy and how are they related?

MATTER is anything that occupies space and can be touched and felt.

ENERGY is anything that used to do "work"

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## KINEMATICS

Kinematics asks how do objects move?

Motion deals with four major measurements:

displacement (d - change of position) - how far does the object move,
time (t) - how long does it take something to move,
velocity (v) - how fast does the object move,
acceleration (a) - how does velocity increase or decrease as the object moves.

Notice there is always some sort of change going on, between a beginning (initial values) and an end (final values). If nothing changes, we are not worried about it (yet).

The definition of velocity is the change of position over time; and acceleration is the change of velocity over time:

$$v = \frac{\Delta x}{\Delta t}$$
$$a = \frac{\Delta v}{\Delta t}$$

The definition of velocity is the derivative of position function over time; and acceleration is derivative of velocity function over time; jerk is derivative of acceleration function over time (PHY 1300 only):

$$\vec{\mathbf{v}} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{x}}}{\Delta t} = \frac{d\,\vec{\mathbf{x}}}{dt}$$
$$\vec{\mathbf{a}} = \frac{d\,\vec{\mathbf{v}}}{dt} \equiv \frac{d^2\vec{\mathbf{x}}}{dt^2}$$
$$\vec{\mathbf{j}} = \frac{d\,\vec{\mathbf{a}}}{dt} \equiv \frac{d^2\vec{\mathbf{v}}}{dt^2} \equiv \frac{d^3\vec{\mathbf{x}}}{dt^3}$$

In Physics, we usually plot position or displacement on a Cartesian coordinate plane using x-y axes. We usually plot velocity on a Cartesian coordinate plane using t-x axes. We usually plot acceleration on a Cartesian coordinate plane using t-v axes.

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## LINEAR MOTION

Objects moving in a straight line undergo linear motion. It is one-dimensional; only horizontal or only vertical.

If acceleration is constant (does not change), the equations relating the measurements of linear motion to each other are called the Galilean Equations, and - without proof - they are:

$$① \quad v_f = v_0 + at$$
$$② \quad x_f = x_0 + v_0 t + \tfrac{1}{2}at^2$$
$$③ \quad v_f^2 = v_0^2 + 2a(x_f - x_0)$$

I always say Physics problems are either definitions problems - do you understand how physicists explain or describe the world - or they are word problems - a real-life situation will be described, hypothetical measurements will be given and you will be asked to compute for other possible measurements.

In solving word problems in Physics, try to keep these points in mind:

• Read the problem very carefully - literally, one word at a time. In English class, you are encouraged to be creative in writing. In Physics, you are expected to be precise and concise - every word is important - creativity applies to ideas, not writing. Exactly what is the question saying - and not saying (you are expected to remember basic information)? Many students misunderstand questions because they rush over the details and definitions. You need to understand the situation described by the question before you can solve for it.

• List the given information and the unknown information in a word problem. Write down the data using the correct symbols, units and abbreviations so that you can refer to your formula sheet and see which one might be helpful.

• Just because an equation says d = something, does not mean we will always use it to solve for d. You are expected to be able to use your algebra skills to manipulate the equation and solve for a different variable, if that is the question. Look at the whole equation, and compare all of the given and unknown information against it.

• Horizontal is NOT the same as vertical. Displacement, velocity, and acceleration are all vectors. For vectors, not only is the size (called the magnitude) of the measurement important, but also the direction. DIRECTION IS IMPORTANT IN PHYSICS! Moving horizontally (left/right) is very different than moving vertically (up/down).

• Sketch a diagram of the situation onto an x-y co-ordinate system. Label it! Make certain to identify the beginning (initial) and the end (final) of your situation. It will help you to better "see" what is supposed to be happening. I recommend to use the origin of x-y plane as initial point. “Begin at the beginning, unless you’re told not to.”

• Mind the positive and negative signs. Never forget that in the Cartesian co-ordinate system (the x-y system) up is positive and down is negative, while left is negative and right is positive. Also, an increase is a positive change, while a decrease is negative. If you want to to do it differently, you are allowed, but you should be consistent. The wrong sign in your data will definitely screw up your calculations and the wrong sign in your answer might indicate something that is not actually supposed to be happening in your problem, or even physically impossible. For horizontal motion: if you have a choice, have your object move to the right - it will minimize negative values in the calculations.

• We are generally dealing only with constant acceleration. For horizontal motion, there is only one acceleration at a time; for vertical motion, the free-fall acceleration (since it is caused by gravity) is the gravitational acceleration: $g = 9.8\ m/s^2$, downward.

• You must show work. In Physics, it is not the answer that is so important, as much as the reasoning, method and calculations used to get your answer. Credit may not be given for a naked answer.

• Remember your units. Physics is entirely a science of measurement. You have nothing without a measurement. Also, Physics is not Mathematics. A number without a unit cannot measure anything and is meaningless. If you are indicating a displacement of ten meters, it is written as d = 10 m, not just as d = 10. A million is irrelevant, but a million dollars is definitely something to be interested in. Additionally, all measurements in Physics have a standard unit agreed to by international treaty. You must use the metric system - all Physics equations assume that you are using the proper unit and they don't work otherwise. If you are given a quantity in a non-standard unit, you must convert.

• Sometimes the algebra turns out to have many, many steps. It is often simpler to plug in your measurements and do arithmetic before the algebra. “Arithmetic is easier than algebra.”

• Use significant figures to indicate precision of the solution. Oftentimes, your calculator will give an answer with eight, nine or even ten digits. Not all of them will be useful, because your data will not have had eight, nine or ten significant figures of precision. Your precision will not often exceed three digits. Round off your answer! Generally speaking, the least precise piece of data determines the precision of the answer.

• Remember that these are idealized situations. In Physics, we use mechanical “models”; accurate but simplified descriptions of reality. At this level, we ignore important factors that would otherwise greatly complicate the calculations - for example: air resistance. Except under carefully controlled conditions, you cannot expect real-life situations to exactly match the calculations done here.

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## FREE-FALL MOTION

Engineless objects moving in a vertical straight line - an object is dropped, or is thrown straight up or down - undergo free-fall motion. It is a special one-dimensional motion where acceleration is gravitational acceleration from the planet you are on.

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\begin{aligned}
g_{Earth} &= 9.8 m/s^2 \text{ (down)} \\
① \quad v_f &= v_0 - gt \\
② \quad y_f &= y_0 + v_0 t - \tfrac{1}{2} g t^2 \\
③ \quad v_f^2 &= v_0^2 - 2g(y_f - y_0)
\end{aligned}
$$

Remember: if you are not on the Earth, gravitational acceleration is not 9.8 m/s². Since, we are not astronauts, this is make-believe, but is perfectly OK for pencil-and-paper problems.

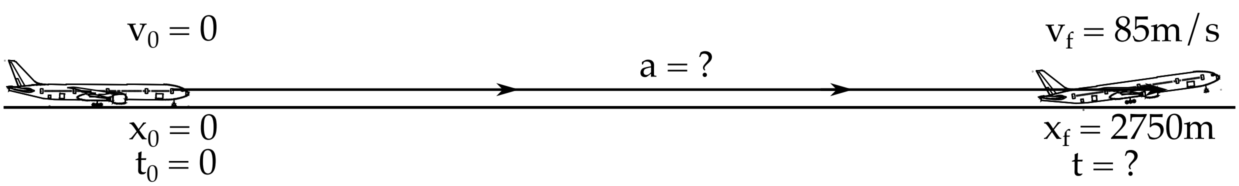
In Math, up-is-positive and down-is-negative - always. This may be inconvenient in Physics. If an object is only falling downward, some Physicists will change to down-is-positive, and avoid negative signs. I don't recommend this.

Remember, free-fall motion does not mean only downward motion. Free-fall motion is a vertical motion; whether up or down.

## Part 2 Linear motion Problems

Problem 1: The largest twin-engine airplane ever made is the Boeing 777. It can carry 375 passengers over 8400 miles at 550 miles per hour. If a Triple -Seven travels 9000 ft (2750 m) down a runway before taking off at a speed of 165 knots (85 m/s), what is its average acceleration and total takeoff time?


Solution:
given: $x_0 = 0,\ x_f = 2750m,\ v_0 = 0,\ v_f = 85m/s$
unknown: $a = ?,\ t = ?$




$$v_f^2 = \cancel{v_0^2} + 2a(x_f - \cancel{x_0})$$

$$\therefore \quad a = \frac{v_f^2}{2x_f}$$

$$a = \frac{(85m/s)^2}{2\,(2750m)}$$

$$a \approx 1.31m/s^2$$

$$v_f = \cancel{v_0} + at$$

$$\therefore \quad t = \frac{v_f}{a}$$

$$t = \frac{85m/s}{1.31m/s^2}$$

$$t \approx 64.9s$$

Problem 2: The Chrysler Pacifica minivan can accelerate from rest through a distance of ¼ mile (402 m) in 15.9 s. What is its average acceleration and its speed at the end the quarter-mile?

Solution:
given: $x_0 = 0,\ x_f = 402m,\ v_0 = 0,\ t = 15.9s$
unknown: $a = ?,\ v_f = ?$

$$x_f = \cancel{x_0} + \cancel{v_0 t} + \frac{1}{2}at^2$$

$$\therefore \quad a = \sqrt{\frac{2x_f}{t^2}}$$

$$a = \sqrt{\frac{2\,(402m)}{\left(15.9s\right)^2}}$$

$$a \approx 3.18m/s^2$$

$$v_f = \cancel{v_0} + at$$

$$v_f = 3.18m/s^2\left(15.9s\right)$$

$$v_f \approx 50.6m/s$$

Problem 3: The Dodge Challenger SRT Hellcat Redeye muscle car can accelerate from rest to 100 mph (44.7 m/s) in 7.4 s. (a) How far does a Challenger travel in this time? A Challenger can brake from 100 mph to a stop in 303 ft (92.4 m). (b) How much time does a Challenger take to stop?

Solution: there are two unrelated parts, with different accelerations
part (a) given: $x_0 = 0,\ v_0 = 0,\ v_f = 44.7m/s,\ t = 7.4s$
part (a) unknown: $a = ?,\ x_f = ?$

$$v_f = \cancel{v_0} + at$$

$$\therefore \quad a = \frac{v_f}{t}$$

$$a = \frac{44.7m/s}{7.4s}$$

$$a \approx 6.04m/s^2$$

$$x_f = \cancel{x_0} + \cancel{v_0 t} + \frac{1}{2}at^2$$

$$x_f = \frac{1}{2}\left(6.04m/s^2\right)(7.4s)^2$$

$$x_f \approx 165m$$

part (b) given: $x_0 = 0,\ v_0 = 44.7,\ x_f = 92.4m,\ v_f = 0$
part (b) unknown: $a = ?,\ t = ?$

$$\cancel{v_f^2} = v_0^2 + 2a(x_f - \cancel{x_0})$$

$$\therefore \quad a = \frac{-\left(v_0^2\right)}{2x_f}$$

$$a = \frac{-\left((44.7m/s)^2\right)}{2\,(92.4m)}$$

$$a \approx -10.8m/s^2$$

$$\cancel{v_f} = v_0 + at$$

$$\therefore \quad t = \frac{-v_0}{a}$$

$$t = \frac{-44.7m/s}{-10.8m/s^2}$$

$$t \approx 4.14s$$

Problem 4: Although the US Navy's 100,000 ton aircraft carriers are huge ships, they are tiny airports, and aircraft cannot takeoff without help. They use 310 ft (94.4 m) catapults to throw aircraft into the air. If a F/A-18 Hornet fighter jet has a takeoff speed of 180 mph (82.7 m/s), what acceleration does the catapult give. What is the launch time? If a Hornet can accelerate at 6.0 m/s² unassisted, what is its unassisted takeoff distance and time?

Solution: Notice there are two different parts: the Hornet taking off using the catapult, versus with its engines only, that have different accelerations.

Part 1: given: $v_0 = 0$, $v_f = 82.7m/s$, $x_0 = 0$, $x_f = 94.4m$
unknown: $a = ?$, $t = ?$

$$v_f^2 = \cancel{v_0^2} + 2a(x_f - \cancel{x_0})$$
$$\therefore \quad a = \frac{v_f^2}{2x_f}$$
$$a = \frac{(82.7m/s)^2}{2\,(94.4m)}$$
$$a \approx 32.2m/s^2$$

$$v_f = \cancel{v_0} + at$$
$$\therefore \quad t = \frac{v_f}{a}$$
$$t = \frac{82.7m/s}{32.2m/s^2}$$
$$t \approx 2.57s$$

Part 2: given: $v_0 = 0$, $v_f = 82.7m/s$, $x_0 = 0$, $a = 6.0m/s^2$
unknown: $x_f = ?$, $t = ?$

$$v_f^2 = \cancel{v_0^2} + 2a(x_f - \cancel{x_0})$$
$$\therefore \quad x_f = \frac{v_f^2}{2a}$$
$$x_f = \frac{(82.7m/s)^2}{2\left(6.0m/s^2\right)}$$
$$x_f \approx 570.m$$

$$v_f = \cancel{v_0} + at$$
$$\therefore \quad t = \frac{v_f}{a}$$
$$t = \frac{82.7m/s}{6.0m/s^2}$$
$$t \approx 13.8s$$

Problem 5: A driver traveling at 10 m/s spots a red light 40 m ahead, and steps on the brake after a one second reaction time. What deceleration would stop the car right at the light?

Solution: this is a two part related motion: reaction time has zero acceleration. The reaction time distance is the initial position for the braking.

Part 1: given: $a_1 = 0,\ v_0 = 10m/s,\ v_1 = 10m/s,\ x_0 = 0,\ t_0 = 0,\ t_1 = 1s$
unknown: $x_1 = ?$

$$\begin{aligned} x_1 &= v_1 t_1 \\ x_1 &= 10m/s\,(1s) \\ x_1 &= 10m \end{aligned}$$

Part 2: given: $v_1 = 10m/s,\ v_f = v_2 = 0,\ x_1 = 10m,\ x_f = x_2 = 30m,\ t_1 = 1s$
unknown: $a_2 = ?$

$$\begin{aligned} \cancel{v_2^2} &= v_1^2 + 2a_2(x_2 - x_1) \\ \therefore \quad a_2 &= \frac{-\left(v_1^2\right)}{2\left(x_2 - x_1\right)} \\ a_2 &= \frac{-\left((10.0m/s)^2\right)}{2\,(40.0m - 10m)} \\ a_2 &\approx -1.67m/s^2 \end{aligned}$$

Problem 6: A speeder zooms past an idling police car at 20 m/s. If the police officer instantly starts chasing the speeder with a constant acceleration of 4 m/s², how long will it be before the police car overtakes the speeder? How far does the speeder travel before being overtaken?

Solution: there are two separate but related motions. You should set up the speeder and police car separately and relate them together because the police car must have the same total time and distance as the speeder.

given: $v_s = 20m/s$, $a_s = 0$, $v_{0p} = 0$, $a_p = 4m/s^2$, $t_s = t_p$, $x_{0s} = x_{0p} = 0$, $x_{fs} = x_{fp}$
unknown: $t_s = t_p = ?$, $x_s = x_p = ?$

speeder (no acceleration): $x_{fs} = v_s t_s$

police car: $x_{fp} = x_{0p} + v_{0p} t_p + \frac{1}{2} a_p t_p^2$

$$
\begin{aligned}
x_{fs} &= x_{fp} \\
v_s t_s &= \cancel{x_{0p}} + \cancel{v_{0p} t_p} + \frac{1}{2} a_p t_p^2 \\
v_s t &= \frac{1}{2} a_p t^2 \\
0 &= \frac{1}{2} a_p t^2 - v_s t \\
0 &= t\left(\frac{1}{2} a_p t - v_s\right) \\
0 &= \frac{1}{2} a_p t - v_s \\
\therefore \quad t &= \frac{2v_s}{a_p} \\
t &= \frac{2\,(20\,m/s)}{4\,m/s^2} \\
t &= 10\,s
\end{aligned}
$$

$$
\begin{aligned}
x_s &= v_s t \\
x_s &= 20m/s\,(10s) \\
x_s &= 200m
\end{aligned}
$$

Problem 7: A speeder zooms past a parked police car at 20 m/s. If the police officer starts chasing the speeder with an acceleration of 4 m/s² after a 2 second reaction time, how long will it be before the police car overtakes the speeder?

Solution: this is also are two separate but related motions. The police car reaction time makes the speeder and police car driving times different. You should set up the speeder and police car separately and relate them together because the police car must still have the same total distance as the speeder.

given: $t_s = t_p + 2s$ because of reaction time

and $v_s = 20m/s,\ a_s = 0,\ v_{0p} = 0,\ a_p = 4m/s^2,\ x_{0s} = x_{0p} = 0,\ x_{fs} = x_{fp}$

unknown: $t_s = ?$

speeder (no acceleration): $x_{fs} = v_s t_s$

police car: $x_{fp} = x_{0p} + v_{0p} t_p + \frac{1}{2} a_p t_p^2$

$$
\begin{aligned}
x_{fs} &= x_{fp} \\
v_s t_s &= \cancel{x_{0p}} + \cancel{v_{0p} t_p} + \frac{1}{2} a_p t_p^2 \\
v_s t_s &= \frac{1}{2} a_p t_p^2 \\
v_s t_s &= \frac{1}{2} a_p \left(t_s - 2\right)^2 \\
v_s t_s &= \frac{1}{2} a_p \left(t_s^2 - 4t_s + 4\right) \\
2v_s t_s &= a_p t_s^2 - 4a_p t_s + 4a_p \\
0 &= a_p t_s^2 - 4a_p t_s - 2v_s t_s + 4a_p \\
0 &= 4t_s^2 - 4(4)t_s - 2(20)t_s + 4(4) \\
0 &= 4t_s^2 - 56t_s + 16 \\
0 &= t_s^2 - 14t_s + 4 \\
\therefore a &= 1,\ b = -14,\ c = 4
\end{aligned}
$$

$$
\begin{aligned}
t_s &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
t_s &= \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(4)}}{2(1)} \\
t_s &= \frac{14 \pm \sqrt{180}}{2} \\
t_s &= \frac{14 \pm 13.42}{2} \\
t_s &= \frac{27.42}{2}, \quad = \frac{0.582}{2} \\
t_s &\approx 13.7s, \quad \cancel{t_s \approx 0.291s}
\end{aligned}
$$

Problem 8: At the climax of the 1933 movie, and the 2005 remake, King Kong fell from the top the 1250 foot (381 m) tall Empire State Building. (a) How long did it take for Kong to land on the street? (b) How fast was he moving as he landed? Ignore air resistance.

Solution: the origin can be chosen as either the top of the building or the street level. Here, street level is chosen.
given: $g = 9.8m/s^2$, $v_0 = 0$, $y_0 = 381m$, $y_f = 0$
unknown: $t = ?$, $v_f = ?$

part (a): There is enough information to use Equation 2 to find the time immediately, but this would require solving a Quadratic Equation. This can be avoided by finding the final velocity, first. (Avoiding the Quadratic Equation is not always possible.)

$$v_f^2 = \cancel{v_0^2} - 2g(\cancel{y_f} - y_0)$$
$$v_f = \sqrt{-2g\left(-y_0\right)}$$
$$v_f = \sqrt{-2\left(9.8m/s^2\right)\left(-381m\right)}$$
$$v_f \approx \pm 86.4m/s$$

Since we know that Kong was falling downward, we want the negative answer.: $v_f \approx -86.4m/s$

part (b):

$$v_f = \cancel{v_0} - gt$$
$$\therefore \; t = \frac{v_f}{-g}$$
$$t = \frac{-86.4m/s}{-9.8m/s^2}$$
$$t \approx 8.8s$$

Problem 9: Standing on the middle of the George Washington Bridge, you drop a rock. Since the rock splashes into the Hudson River after 3.72 s, how high is the bridge? How fast is the rock moving as it hits the water? Ignore air resistance. (212+10ft≈67.7m)

Solution: the origin can be chosen as either the bridge level or the river level. Here, bridge level is chosen.
given: $g = 9.8m/s^2$, $y_0 = 0$, $v_0 = 0$, $t = 3.72s$
unknown: $h_{bridge} = \left|y_{rock}\right| = ?$, $v_f = ?$

$$y_{rock} = \cancel{y_0} + \cancel{v_0 t} - \frac{1}{2} g t_{rock}^2$$
$$y_{rock} = -\frac{1}{2}\left(9.8m/s^2\right)(3.72s)^2$$
$$y_{rock} \approx -67.8m$$
$$\therefore h_{bridge} \approx 67.8m$$

$$v_f = \cancel{v_0} - gt$$
$$v_f = -9.8m/s^2\,(3.72s)$$
$$v_f \approx -36.5m/s$$

Problem 10: Standing on the middle of the Verrazano Bridge on a foggy day, you cannot see the water. However, when you drop a rock, you hear the sound of the rock splashing into The Narrows 4.06 s later. How high is the bridge? Ignore air resistance; the constant speed of sound is 343 m/s. (228+10ft ≈ 72.5m) t-sound=72.5/343=0.211s; t-rock=3.847s

Solution: this is a two part motion: the rock falls down, the sound returns up.

given: $g = 9.8m/s^2,\ y_0 = 0,\ v_{0\ rock} = 0,\ v_{sound} = 343m/s,$
$t_{rock} + t_{sound} = 4.06s,\ -y_{rock} = y_{sound}$

unknown: $h_{bridge} = \left|y_{rock}\right| = ?,\ v_f = ?$

$$\begin{aligned}
-y_{rock} &= y_{sound} \\
-\left(\cancel{y_0} + \cancel{v_0 t} - \tfrac{1}{2} g t_{rock}^2\right) &= v_{sound}\ t_{sound} \\
\tfrac{1}{2} g t_{rock}^2 &= v_{sound}\left(4.06 - t_{rock}\right) \\
\therefore\ 4.9 t_{rock}^2 + 343 t_{rock} - 1393 &= 0 \\
t_{rock}^2 + 70 t_{rock} - 284.2 &= 0 \\
\therefore a = 1,\ b = 70,\ c &= -284.2
\end{aligned}$$

$$t_{rock} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t_{rock} = \frac{-(70) \pm \sqrt{(-70)^2 - 4(1)(-284.2)}}{2(1)}$$

$$t_{rock} = \frac{-70 \pm \sqrt{6037}}{2}$$

$$t_{rock} = \frac{-70 \pm 77.70}{2}$$

$$t_{rock} = \frac{7.70}{2},\quad = \cancel{\frac{-147.70}{2}}$$

$$t_{rock} \approx 3.85s$$

$$y_{rock} = -\tfrac{1}{2} g t_{rock}^2$$

$$y_{rock} = -\tfrac{1}{2}\left(9.8m/s^2\right)\left(3.85s\right)^2$$

$$y_{rock} \approx -72.6m$$

$$\therefore h_{bridge} \approx 72.6m$$

Problem 11: A batter hits a baseball from 1 m above the ground at 35m/s straight upward. It is a high pop up. How high does the ball rise above the ground? If the ball is allowed to drop to the ground, how long would it take and how fast is it moving when it hits?

Solution: ground level is chosen as the origin
given: $v_0 = 35m/s,\ v_{top} = 0,\ y_0 = 1m,\ y_f = 0,\ g = 9.8m/s^2$

unknown: $y_{top} = ?,\ v_f = ?$

$$\begin{aligned}
v_f^2 &= v_0^2 - 2g(y_f - y_0) \\
\cancel{v_{top}^2} &= v_0^2 - 2g(y_{top} - y_0) \\
0 &= v_0^2 - 2g(y_{top} - y_0) \\
2g(y_{top} - y_0) &= v_0^2 \\
y_{top} - y_0 &= \frac{v_0^2}{2g} \\
\therefore\ y_{top} &= \frac{v_0^2}{2g} + y_0 \\
y_{top} &= \frac{\left(35m/s\right)^2}{2\left(9.8m/s^2\right)} + 1m \\
y_{top} &= 63.5m
\end{aligned}$$

The algebra is a bit complicated, and so you might want to simplify by plugging in the data and doing some arithmetic first. “Arithmetic is easier than algebra.”

Problem 12: You toss a ball straight upward at 11 m/s to a friend, who catches it on a balcony 6 m above you. How fast is the ball moving when caught, and how long will take to be caught?

Solution: given: $v_0 = 10m/s$, $y_0 = 0$, $y_f = 6m$, $g = 9.8m/s^2$

unknown: $v_f = ?$, $t = ?$

$$v_f^2 = v_0^2 - 2g(y_f - \cancel{y_0})$$
$$v_f = \sqrt{v_0^2 - 2gy_f}$$
$$v_f = \sqrt{(11m/s)^2 - 2\left(9.8m/s^2\right)\left(6m\right)}$$
$$v_f \approx \pm 1.84m/s$$

Note: there are two answers, depending on whether the ball is caught on the way up (the positive velocity) or is falling back down (the negative velocity) when caught. The question is not specific, and so both velocities are correct. Both velocities must be used to determine two time values.

$$v_f = v_0 - gt$$
$$\therefore \; t = \frac{v_f - v_0}{-g}$$

$$t_{pos} = \frac{1.84m/s - 11m/s}{-9.8m/s^2}$$
$$t_{pos} \approx 0.935s$$

$$t_{neg} = \frac{-1.84m/s - 11m/s}{-9.8m/s^2}$$
$$t_{neg} \approx 1.40s$$

# Part 3: Vectors and 2-dimensional motion - projectiles

**"Nothing is too wonderful to be true." attributed to Michael Faraday (1791 – 1867)**

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## VECTOR vs. SCALAR measurements

Scientists measure everything. In Physics, many measurements are vectors, others are scalars.

Vectors have MAGNITUDE (how large is the measurement) and DIRECTION (which way does the measurement go). In Physics, the direction is often important. For example: Imagine a tourist comes up to you in Herald Square and asks "Where's Times Square?" Your answer would be "Go eight blocks north on Broadway." In Physics, this would be a displacement vector with magnitude "eight blocks" and direction "north."

> Important vectors in PHY 1100/1300 include displacement, velocity, acceleration, force (including weight), momentum and torque.

Scalars have MAGNITUDE ONLY: In Physics, sometimes only a magnitude is relevant. For example, a weather report would say "The high temperature today is 68°F," not 68°F north.

> Important scalars in PHY 1100/1300 include distance, speed, time, mass, work and energy, density, pressure and temperature.

We draw vectors as arrows on the x-y plane. We point the arrow in the direction of the vector and the length of the arrow indicates the magnitude. We will follow the CONVENTIONAL ANGLE measurement from Math class - the positive x-axis is 0°, moving counterclockwise is a positive angle and moving clockwise is negative.

Pay attention to the relationship between vectors and conventional angles:

$$\vec{A} = 60m, \quad \theta_A = 40^\circ$$

the basic trigonometry of the right triangle: (SOHCAHTOA):

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

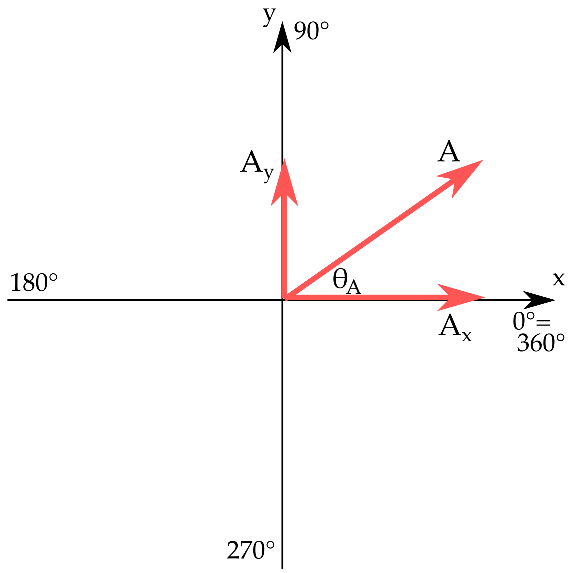
and therefore a vector has horizontal and vertical components:

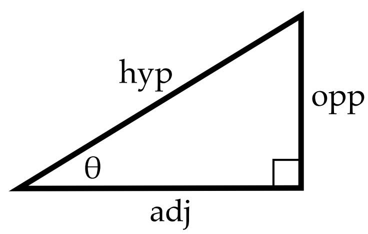
$$A_x = \left|A\right| \cos\theta_A \qquad A_y = \left|A\right| \sin\theta_A$$

The vector's magnitude and direction can be recovered from its components.

$$\left|A\right| = \sqrt{A_x^2 + A_y^2} \qquad \theta_A = \arctan\left|\frac{A_y}{A_x}\right| \equiv \tan^{-1}\left|\frac{A_y}{A_x}\right|$$

because all vector calculations are done on the components, not the magnitude or direction.




For 3-dimensional vectors in the x-y-z space, the magnitude and direction of vectors is usually replaced by ordered triplet or i-j-k notation, using the components directly. (PHY 1300 only)

$$\vec{A} \equiv (A_x, A_y, A_z) \quad \text{or} \quad \vec{A} \equiv A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

---

## Vector addition/subtraction

Vector math is different than scalar math. Two plus two may not equal four. All vector calculations are done on the components, not on the magnitudes directly.

To add or subtract vectors, find both sets of vector components, add the components separately and put the resultant components back together:

if $\vec{A} + \vec{B} = \vec{R}$,

then $A_x + B_x = R_x = \left|A\right| \cos\theta_A + \left|B\right| \cos\theta_B$

and $A_y + B_y = R_y = \left|A\right| \sin\theta_A + \left|B\right| \sin\theta_B$

therefore $\left|R\right| = \sqrt{R_x^2 + R_y^2}$ $\qquad\qquad \theta_R = \tan^{-1}\left|\dfrac{R_y}{R_x}\right|$

---

## Vector multiplication

(PHY 1300 only)

There are two different types of vector multiplication depending the product.

Vector times vector equals scalar is called the VECTOR DOT PRODUCT; also called the scalar product or inner product:

$$\vec{A} \bullet \vec{B} = R = \left|A\right| \left|B\right| \cos\phi_{AB}$$

$$\vec{A} \bullet \vec{B} = R = A_x B_x + A_y B_y + A_z B_z$$

Vector times vector equals vector is called the VECTOR CROSS PRODUCT; also called the vector product or outer product.

The cross product is formally done as a matrix multiplication:

$$\begin{aligned}
\vec{A} \times \vec{B} &= \vec{R} \\
&= \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix} \\
&\equiv \begin{bmatrix} A_y & A_z \\ B_y & B_z \end{bmatrix} \hat{i} \; - \; \begin{bmatrix} A_x & A_z \\ B_x & B_z \end{bmatrix} \hat{j} \; + \; \begin{bmatrix} A_x & A_y \\ B_x & B_y \end{bmatrix} \hat{k} \\
&\equiv \left(A_y B_z - A_z B_y\right) \hat{i} \; - \; \left(A_x B_z - A_z B_x\right) \hat{j} \; + \; \left(A_x B_y - A_y B_x\right) \hat{k}
\end{aligned}$$

Note: the cross product is not commutative

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

the cross product is anti-commutative

$$\vec{A} \times \vec{B} = -\left(\vec{B} \times \vec{A}\right)$$

Since matrix multiplication is tedious, there is a shortcut for the 3-dimensional (ONLY) cross product:

vector cross product magnitude-only shortcut:

$$\left|\vec{A} \times \vec{B}\right| = \left|\vec{R}\right| = \left|A\right| \left|B\right| \sin\phi_{AB}$$

vector cross product direction-only shortcut (the right-hand-rule, RHR):

> Hold your right hand with fingers straight and thumb out;
> step 1: point your fingers in the direction of vector A,
> step 2: point your thumb in the direction of vector B,
> step 3: your palm faces the the direction of vector R.
> Note: there are alternate ways to set up the RHR.

Note: the direction of the vector cross product resultant R is always perpendicular to the plane formed by the original two vectors A and B. If A and B are on the x-y plane, R is along the z-axis. The three vectors exist in a three-dimensional space.

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## PROJECTILE MOTION

Two-dimensional motion is motion that is not strictly on a straight line. Rather it combines up/down motion with left/right motion.

Projectile motion is most typical of two dimensional motion: an object is thrown into the air, but not straight up or down, so that as gravity pulls the object downward, it also moves horizontally.

Therefore, the initial velocity vector is actually a total of the initial horizontal and initial vertical velocities. Computing with the initial velocity vector is not really feasible.

Instead, the initial velocity must be broken up as a vector quantity into its horizontal and vertical components: $v_{0x}$ and $v_{0y}$. These individual components are used to solve the problem. In other words, there are separate horizontal calculations and vertical calculations in any vector problem. The only information that may transfer between the two parts is the time.

At this point, you should review the material on what vectors are and how they are used, as well as make sure that you can use basic trigonometry (trigonometry is the mathematics of angles - the direction of vectors is usually measured as an angle).

| **horizontal (x-direction)** | **vertical (y-direction)** |
|---|---|
| $v_{0x} = v_0 \cos\theta_0$ | $v_{0y} = v_0 \sin\theta_0$ |
| no acceleration: $a_x = 0$ | $a_y = g_{Earth} = 9.8 m/s^2$ (down) |
| velocity is constant: $v_{0x} = v_{fx}$ | ① $v_{fy} = v_{0y} - gt$ |
| $x = v_{0x} t$ | ② $y_f = y_0 + v_{0y} t - \frac{1}{2} g t^2$ |
| | ③ $v_{fy}^2 = v_{0y}^2 - 2g(y_f - y_0)$ |

Remember: if you are not on the Earth, gravitational acceleration is not 9.8 $m/s^2$. Since, we are not astronauts, this is make-believe, but is perfectly OK for pencil-and-paper problems.

Strictly speaking, the 2-dimension position function - called the the trajectory (derivable from the horizontal equation and vertical equation 2) is:

$$y(x) = y_0 + \tan\theta_0\, x - \frac{g}{2v_0^2\cos^2\theta_0} x^2$$

Notice this a parabolic function.

There are other derived equations, such as the Range Formula. As a general rule, you are not supposed to use derived equations, unless you can derive them from the basic equations.

# Part 3 2-dimensional motion Problems

Problem 1: A place kicker in the National Football League always tries for a "hang time" of over four seconds on the kick-off. If a football is kicked upward at 26m/s and an angle of 60°, (a) how long will the ball remain in the air before hitting the ground? (b) How high will the ball rise? (c) How far away will the ball land? Ignore air resistance.

Solution:
given: $v_0 = 26m,\ \theta_0 = 60^{\circ},\ x_0 = 0,\ y_0 = 0,\ y_f = 0,\ v_{top\,y} = 0,\ v_{fy} = -\,v_{0y}$
unknown: $t_{tot} = ?,\ y_{top} = ?,\ x_{tot} = ?$

$$v_{0x} = v_0 \cos\theta_0 = 26m/s\left(\cos 60^\circ\right) = 13m/s$$

$$v_{0y} = v_0 \sin\theta_0 = 26m/s\left(\sin 60^\circ\right) \approx 22.5m/s$$

(a)

$$\begin{aligned} v_{fy} &= v_{0y} - gt \\ \therefore\ t_{top} &= \frac{\cancel{v_{top\,y}} - v_{0y}}{-g} \\ t_{top} &\approx \frac{-22.5m/s}{-9.8m/s^2} \\ t_{top} &\approx 2.3s \\ t_{total} &= 2t_{top} \approx 2\,(2.3s) \\ t_{total} &\approx 4.6s \end{aligned}$$

OR

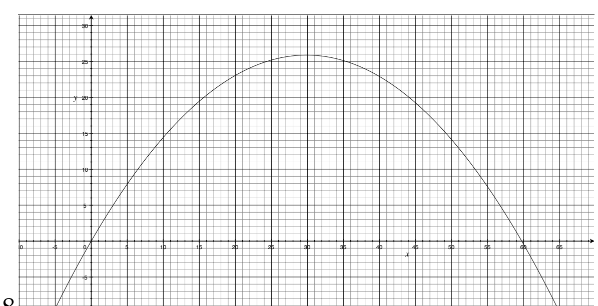
$$\begin{aligned} v_{fy} &= v_{0y} - gt \\ -v_{0y} &= v_{0y} - gt_{total} \\ \therefore\ t_{total} &= \frac{-2v_{0y}}{-g} \\ t_{total} &\approx \frac{2\left(-22.5m/s\right)}{-9.8m/s^2} \\ t_{total} &\approx 4.6s \end{aligned}$$

(b)

$$\begin{aligned} y_f &= y_0 + v_{0y}\,t - \tfrac{1}{2}gt^2 \\ y_{top} &= \cancel{y_0} + v_{0y}\,t_{top} - \tfrac{1}{2}gt_{top}^2 \\ y_{top} &\approx 22.5m/s\,(2.3s) - \tfrac{1}{2}\left(9.8m/s^2\right)(2.3s)^2 \\ y_{top} &\approx 51.8m - 25.9m \\ y_{top} &\approx 25.9m \end{aligned}$$

(c)

$$\begin{aligned} x &= v_{0x}t \\ x_{tot} &\approx 13m/s\left(4.6s\right) \\ x_{tot} &\approx 59.8m \end{aligned}$$



Problem 2: Before the invention of gunpowder cannon, the most powerful weapon in the world was the trebuchet, a sling, lever and counterweight machine that could smash castle walls by hurling quarter-ton boulders the length of three football fields. If a trebuchet throws a boulder from a height of 20 m, at a speed of 60 m/s, upward at 40°, (a) how high will the boulder go, (b) how long is it in the air, (c) how far will it go before hitting the ground, and (d) what is its velocity (magnitude and direction) as it hits the ground? Ignore air resistance.

Solution:
given: $v_0 = 60m/s$, $\theta_0 = 40^\circ$, $x_0 = 0$, $y_0 = 20m$, $y_f = 0$, $v_{top\,y} = 0$, $v_{fy} \neq -v_{0y}$
unknown: $y_{top} = ?$, $t_{total}$, $x_{total} = ?$, $v_f = ?$, $\theta_f = ?$

$$v_{0x} = v_0 \cos\theta_0 = 60m/s\left(\cos 40^\circ\right) \approx 45.96m/s$$

$$v_{0y} = v_0 \sin\theta_0 = 60m/s\left(\sin 40^\circ\right) \approx 38.57m/s$$

part (a):

$$\begin{aligned} v_{fy}^2 &= v_{0y}^2 - 2g(y_f - y_0) \\ \cancel{v_{top\,y}^2} &= v_{0y}^2 - 2g(y_{top} - y_0) \\ 0 &= v_{0y}^2 - 2g(y_{top} - y_0) \\ \therefore\ y_{top} &= \frac{v_{0y}^2}{2g} + y_0 \\ y_{top} &= \frac{(38.57m/s)^2}{2\left(9.8m/s^2\right)} + 20m \\ y_{top} &= 95.7m \end{aligned}$$

part (b):

$$\begin{aligned} &\cancel{y_f} = y_0 + v_{0y}\,t - \tfrac{1}{2}gt^2 \\ &0 = 20m + (38.57m/s)\,t_{total} - \tfrac{1}{2}\left(9.8m/s^2\right)t_{total}^2 \\ &0 = -4.9t_{total}^2 + 38.57t_{total} + 20 \\ &a = -4.9,\ b = 38.57,\ c = 20 \end{aligned}$$

$$t_{total} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t_{total} = \frac{-(38.57) \pm \sqrt{(38.57)^2 - 4(-4.9)(20)}}{2(-4.9)}$$

$$t_{total} = \frac{-38.57 \pm \sqrt{1880}}{-9.8}$$

$$t_{total} = \frac{-38.57 \pm 43.35}{-9.8}$$

$$t_{total} = \cancel{\frac{4.78}{-9.8}}, \quad = \frac{-81.92}{-9.8}$$

$$t_{total} \approx 8.36s$$

part (c):

$$x_{total} = v_{0x} t_{total}$$

$$x_{total} = 45.96m/s\,(8.36s)$$

$$x_{total} \approx 384m$$

part (d):

$$v_{fx} = v_{0x} \approx 45.96m/s$$

$$v_{fy} = v_{0y} - gt$$

$$v_{fy} = 38.57m/s - 9.8m/s^2\,(8.36s)$$

$$v_{fy} \approx -43.36m/s$$

$$v_f = \sqrt{v_{fx}^2 + v_{fy}^2}$$

$$v_f = \sqrt{(45.96)^2 + (-43.36)^2}$$

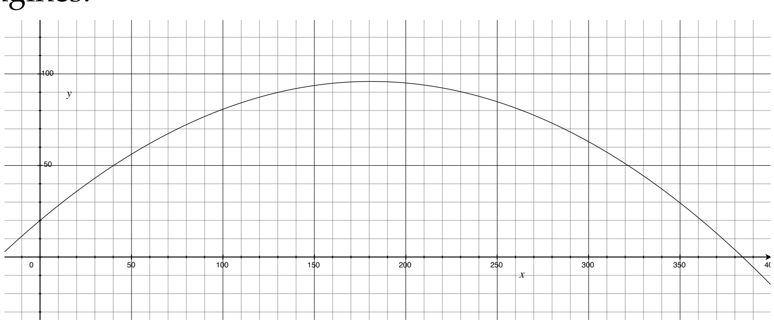
$$v_f \approx 63.2m/s$$

$$\theta_f = \tan^{-1}\left(\frac{v_{fy}}{v_{fx}}\right)$$

$$\theta_f = \tan^{-1}\left(\frac{-43.36}{45.96}\right)$$

$$\theta_f \approx -43.3^\circ$$

By the way, the word "engineer" originally meant a person who could design and build such "military engines."



Problem 3: A batter hits a baseball 1 m above the ground at 45 m/s upward at a 70° angle. The 3 m tall centerfield wall is 130 m away. It is a long fly ball, but is it a home run - does it clear the wall, or does it hit the wall? Ignore air resistance.

Solution:
given: $v_0 = 45m/s$, $\theta_0 = 70°$, $x_0 = 0m$, $y_0 = 1m$, $y_f > 3m$, $x_{tot} = 130m$, $v_{fy} \neq -v_{0y}$
unknown: $y_f = ?$, $t_{tot} = ?$

$$x = v_{0x}t$$
$$\frac{x}{v_{0x}} = t$$
$$\therefore \; t = \frac{x}{v_0 \cos\theta_0}$$
$$t_{130m} = \frac{130m}{45m/s \;\; \cos 70°}$$
$$t_{130m} \approx 8.447s$$

$$y_f = y_0 + v_{0y}t - \frac{1}{2}gt^2$$
$$y_{130m} = 1m + 45m/s\left(\sin 70^{\circ}\right)(8.447s) - \frac{1}{2}\left(9.8m/s^2\right)(8.447s)^2$$
$$y_{130m} \approx 1m + 357.2m - 349.6m$$
$$y_{130m} \approx 8.6m$$

YES; the ball will clear the wall by 5.6 m (= 8.6m - 3m).

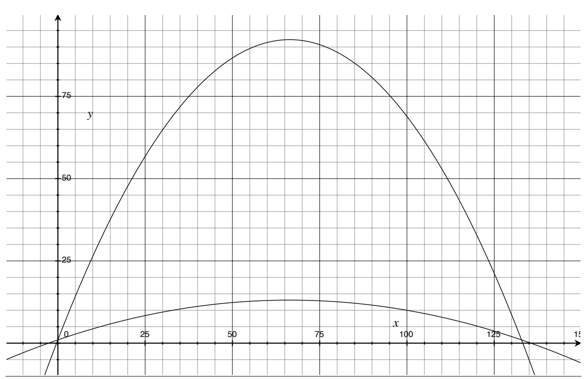
Problem 4: A batter hits a baseball 1 m above the ground at 45m/s upward at a 20° angle. The 3 m tall centerfield wall is 130m away. It is a long line drive, but is it a home run - does it clear the wall, or does it hit the wall? Ignore air resistance.

Solution:
given: $v_0 = 45m/s$, $\theta_0 = 20°$, $x_0 = 0m$, $y_0 = 1m$, $x_{tot} = 130m$, $v_{fy} \neq -v_{0y}$
unknown: $y_f = ? > 3m$, $t_{tot} = ?$

$$x = v_{0x}t$$
$$\frac{x}{v_{0x}} = t$$
$$\therefore\ t = \frac{x}{v_0 \cos\theta_0}$$
$$t_{130m} = \frac{130m}{45m/s\ \ \cos 20°}$$
$$t_{130m} \approx 3.074s$$

$$y_f = y_0 + v_{0y}t - \frac{1}{2}gt^2$$
$$y_{130m} = 1m + 45m/s\left(\sin 20^{\circ}\right)(3.074s) - \frac{1}{2}\left(9.8m/s^2\right)(3.074s)^2$$
$$y_{130m} \approx 1m + 47.3m - 46.3m$$
$$y_{130m} \approx 2.0m$$

NO; the ball will hit the wall by 1.0 m below the top (2.0m - 3m).



Problem 5: What must be the initial speed of a basketball jump shot, if it is thrown from a height of 2.1 m, at an angle of 50° so that the ball swishes into the 3.0 m high basket, 5.2 m away?

Solution:
given: $\theta_0 = 50°,\ x_0 = 0m,\ y_0 = 2.1m,\ x_f = 5.2m,\ y_f = 3.0m,\ v_{fy} \neq -v_{0y}$
unknown: $v_0 = ?$

$$y_f = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

$$y_f = y_0 + v_0 \sin\theta_0\, t - \frac{1}{2}gt^2$$

$$y_f = y_0 + \cancel{v_0} \sin\theta_0 \frac{x}{\cancel{v_0}\cos\theta_0} - \frac{g}{2}\left(\frac{x}{v_0\cos\theta_0}\right)^2$$

$$y_f = y_0 + \tan\theta_0\ x - \frac{gx^2}{2v_0^2\cos^2\theta_0}$$

$$x = v_{0x}t$$

$$\frac{x}{v_{0x}} = t$$

$$\therefore\ t = \frac{x}{v_0\cos\theta_0}$$

$$3.0 = 2.1 + \tan 50^{\circ}\,(5.2) - \frac{9.8\,(5.2)^2}{2v_0^2\cos^2 50^{\circ}}$$

$$3.0 \approx 2.1 + 6.2 - \frac{265}{0.826v_0^2}$$

$$3.0 \approx 8.3 - \frac{321}{v_0^2}$$

$$-5.3 \approx -\frac{321}{v_0^2}$$

$$\therefore\ v_0 \approx \sqrt{\frac{321}{5.3}}$$

$$v_0 \approx 7.9m/s$$

Notice that this question is asking for INITIAL speed. The question is sort of "backwards." This tends to complicate the Math and I did not complete the algebra, before plugging in the data. I simplified by doing some arithmetic first. "Arithmetic is easier than algebra."

Problem 6: The US Navy's 9700 ton World War II-era cruiser, USS Brooklyn (built at the New York [Brooklyn] Navy Yard), was armed with fifteen 6 inch/47 caliber cannon that fired 130 lb shells at a velocity of 2500 ft/s (762 m/s) to a range of 26,100 yards (14.8 miles; 23,900 m). What angle of elevation did Brooklyn fire her guns to reach this range? Ignore air resistance. Hint: $2\sin\theta\cos\theta = \sin 2\theta$. (PHY 1300 only)

Solution: given: $\left|v_0\right| = 762 m/s,\ x_f = 23{,}900 m,\ y_0 = y_f = 0,\ v_{fy} = -v_{0y} = -v_0 \sin\theta_0$
unknown: $\theta_0 = ?$

$$x = v_{0x} t$$
$$x = v_0 \cos\theta_0\ t$$
$$\therefore\ t = \frac{x}{v_0 \cos\theta_0}$$

$$and \quad v_{fy} = v_{0y} - gt$$
$$-v_{0y} = v_{0y} - gt$$
$$2v_0 \sin\theta_0 = gt$$
$$\therefore\ t = \frac{2v_0 \sin\theta_0}{g}$$

$$combined \quad \frac{x}{v_0\cos\theta_0} = \frac{2v_0\sin\theta_0}{g}$$
$$x = \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g}$$
$$x = \frac{v_0^2 \sin 2\theta_0}{g}$$

$$\therefore\ \theta_0 = \frac{1}{2}\sin^{-1}\left(\frac{xg}{v_0^2}\right)$$
$$\theta_0 = \frac{1}{2}\sin^{-1}\left(\frac{23{,}900m\,(9.8m/s^2)}{(762m/s)^2}\right)$$
$$\theta_0 \approx 11.9^\circ$$

Note: Brooklyn actually fired at 47.5° for maximum range, because of air resistance.

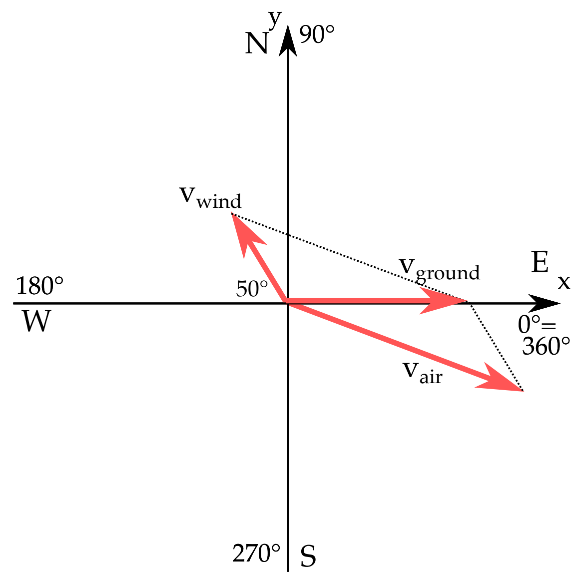
Problem 7: An airliner traveling from Chicago to New York with an airspeed of 250 m/s needs to fly due east, but encounters a steady 20 m/s crosswind blowing at 50° north of west. What heading should the airliner take and what would be its groundspeed?

Solution: given and unknown:

$$\left|v_{air}\right| = 250\ m/s,\ \theta_{air} = ?$$

$$\left|v_{wind}\right| = 20\ m/s,\ \theta_{wind} = 130^{\circ}$$

$$\left|v_{ground}\right| = ?\ ,\ \theta_{ground} = 0^{\circ}$$




$$\vec{v}_{ground} = \vec{v}_{air} + \vec{v}_{wind}$$

Beginning with horizontal components:

$$v_{ground\ x} = v_{air\ x} + v_{wind\ x}$$

$$\left|v_{ground}\right| \cos\theta_{ground} = \left|v_{air}\right| \cos\theta_{air} + \left|v_{wind}\right| \cos\theta_{wind}$$

$$\left|v_{ground}\right| \cos 0^{\circ} = \left(250\ m/s\right) \cos\theta_{air} + (20\ m/s)\cos 130^{\circ}$$

The $v_{ground}$ and $\theta_{air}$ are both unknown. Check vertical components to see if the situation simplifies.

$$v_{ground\ y} = v_{air\ y} + v_{wind\ y}$$

$$\left|v_{ground}\right| \sin\theta_{ground} = \left|v_{air}\right| \sin\theta_{air} + \left|v_{wind}\right| \sin\theta_{wind}$$

$$\left|v_{ground}\right| \sin 0^{\circ} = \left(250\ m/s\right) \sin\theta_{air} + (20\ m/s)\sin 130^{\circ}$$

$$0 = \left(250\ m/s\right) \sin\theta_{air} + 15.3\ m/s$$

$$\therefore\ \theta_{air} = \sin^{-1}\left(\frac{-15.3\ m/s}{250\ m/s}\right)$$

$$\theta_{air} \approx -3.50^{\circ}$$

Now, bring $\theta_{air}$ back to horizontal components.

$$\therefore \left|v_{ground}\right| \cancel{\cos 0^{\circ}} = \left(250\ m/s\right)\cos\left(-3.50^{\circ}\right) + \left(20\ m/s\right)\cos 130^{\circ}$$

$$\left|v_{ground}\right| = 249.5\ m + \left(-12.9\ m/s\right)$$

$$\left|v_{ground}\right| \approx 236.6 m/s$$

# Part 4: Newton's Laws of Motion

**"I do not know how others might see me, but to myself, I seem to have been merely a child playing on the seashore, diverting myself in now and then finding a pebble more smooth or a shell more beautiful than others, whilst before me the great ocean of Truth lay all undiscovered." Isaac Newton (1642 – 1727)**

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## MECHANICS

Mechanics asks why do objects move?

Newton's Laws of Motion

The First Law - the law of equilibrium - an object at rest will stay at rest, on object in motion will stay in motion - no net force = no acceleration (change of motion).

The Second Law - the law of inertia; "force causes acceleration" - yes net force = yes acceleration.

$$\sum \vec{F} = m\vec{a}$$

the sum of force vectors (acting on a mass) equals the mass times its acceleration

This is the important law, because we be using it a lot for problem solving.

The Third Law - the law of action-reaction - for every action, there is an equal but opposite reaction - if any force is applied to an object, that object tends to resist with a opposite force.

Be careful: weight and mass are not the same thing in Physics: weight is the force of gravity; mass measures "inertia" - resistance to change of motion.

Remember: force is a vector, and just like any other vector, direction is always important, and horizontal is always different from vertical.

$$\sum F_x = ma_x$$
$$\sum F_y = ma_y$$

The right side of Newton's Second Law is mathematically simple, it is a linear equation (a simple multiplication). It is the left side that is difficult, because it is an indeterminate sum. The forces must be identified before they can be added - as vectors. This is the important part of the Physics, the rest is “just Math”.

Forces are identified on a free-body diagram. A free-body diagram reduces a mass to a point at the origin of an x-y co-ordinate plane, and draws all the forces (separated into vector components as necessary) acting on the object as arrows pulling out from that point. Make the diagram neat, to reasonable scale and fairly large, so that you can write on and read off your information. Often, the forces can be determined and we need to find the acceleration. With the acceleration in hand, we can use it in a motion problem.

There are four major forces that you should check for:

1) EXTERNAL FORCES are any forces that are not a property of the object itself. They are applied to the object from outside of the object.

2) WEIGHT is the force of gravity. Weight makes things fall downward. Weight and mass are not the same thing. Weight is the gravitational force that goes downward toward the center of the Earth. Mass measures inertia. All objects in PHY 1100/1300 have mass, even if it is not given. Note, there are objects in other PHY classes that do not have mass.

$$weight = mg$$

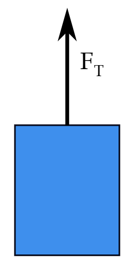
3) NORMAL FORCES are an important action-reaction force. They are contact forces - the object must touch a surface for a normal force to exist. They are perpendicular to the contact surface. There is now simple formula for normal force; it is determined by analyzing the free-body diagram.

4) FRICTION FORCES are contact sliding resistance forces, because surfaces are not perfectly smooth/slippery. Real surfaces are rough. Friction direction is opposite to motion direction. Static friction means not moving yet/trying to start moving. Kinetic friction means already moving. The coefficient of friction - $\mu$ (read “mu”) - is the experimentally measured surface “roughness” and determines maximum friction force. You are not expected to memorize coefficient values. Note: there is always friction in real life - nothing is perfectly slippery. However, in these classes, if the problem does not specifically say the magic word friction, we will pretend that there is no friction.

$$F_{fr} \leq \mu F_N$$

If there is more than one mass, there needs to be more than one free-body diagram. Each mass has its own set of forces. Each diagram is then applied to its own Newton’s Second Law calculation. In real life, real things are not made of one piece. They may have many thousands of parts. Each must be Physically analyzed and Mathematically calculated individually. In real life engineering, this is done mostly on computers. Computer aided design/engineering/manufacturing software is standard in modern engineering.

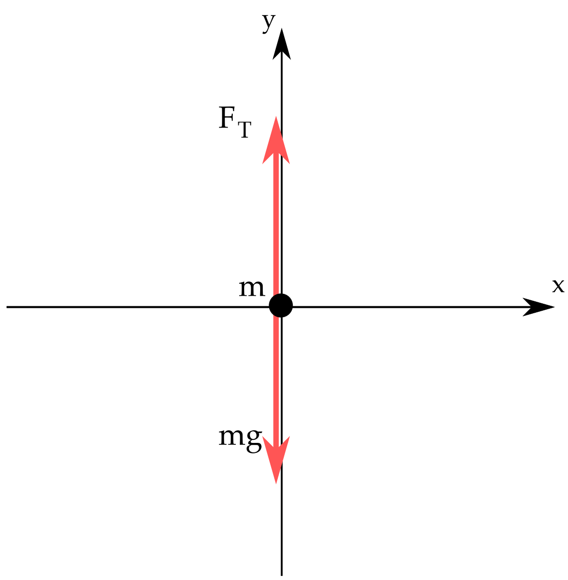
Example: A 2000 kg elevator is pulled upward by a cable. If the tension force is 22,000 N, what is the elevator's acceleration? Ignore air resistance?




Solution:
given: $m = 2000kg,\ F_T = 22{,}000N$

unknown: $a = ?$

Free-body diagram




Newton's Second Law

$$\sum F_y = ma_y$$
$$F_T - mg = ma$$
$$\therefore a = \frac{F_T - mg}{m}$$
$$a = \frac{22{,}000N - 2000kg\left(9.8m/s^2\right)}{2000kg}$$
$$a = 1.2m/s^2$$

INCLINED PLANE is a fancy term for a ramp. Skiing down a mountainside or pushing a cart up a ramp are typical inclined plane problems. For students, the most unusual part of inclined planes is the free-body diagram, where the x-y plane is not horizontal and vertical. "The only reason why up is positive and down is negative; why right is positive and left is negative, is because Rene Descartes said so 400 years ago. If that's not convenient, don't do it that way." It's not convenient for an inclined plane. Rotate the axes, so that the x-axis follows the incline.

A PULLEY makes the motion direction of two connected masses different. Therefore, the x-y plane is not automatically up-is-positive and down-is-negative. "The only reason why up-is-positive and down-is-negative; why right-is-positive and left-is-negative, is because Rene Descartes said so 400 years ago. If that’s not convenient, don’t do it that way.” Let positive follow the direction of motion.

# Part 4 Newton's Laws Problems

Problem 1: If a 50 kg mass experiences a net force of $\vec{F} = (150N)\hat{i} + (100N)\hat{j}$, what is the acceleration of the mass? (PHY 1300 only.)

Solution:
given: $m = 50kg$, $F = (150N)\hat{i} + (100N)\hat{j}$
unknown: $a = ?$

$$\sum \vec{F} = m\vec{a}$$

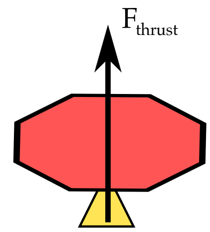
$$\therefore \quad \vec{a} = \frac{\vec{F}}{m}$$

$$\vec{a} = \frac{(150N)\hat{i} + (100N)\hat{j}}{m}$$

$$\vec{a} = \frac{150N}{50kg}\hat{i} + \frac{100N}{50kg}\hat{j}$$

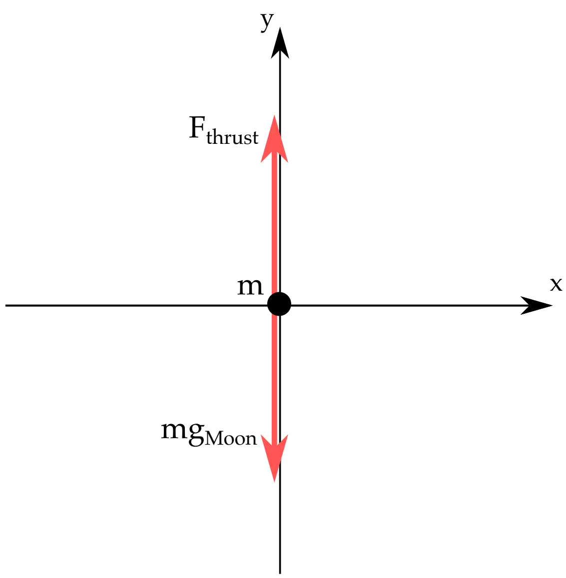
$$\vec{a} = \left(3m/s^2\right)\hat{i} + \left(2m/s^2\right)\hat{j}$$

Problem 2: The only crewed spacecraft designed to operate solely in deep space (not near the Earth) was Lunar Module, built by Grumman Aircraft of Bethpage, Long Island, that took the Apollo astronauts down and back up from the Moon. On liftoff from the Moon, the ascent stage had a mass of 4550 kg. Since the ascent rocket produced an upward thrust force of 15,600 N, what was the ascent stage's acceleration? ($g_{Moon} = 1.6$ m/s²)




Solution:
given: $m = 4550kg$, $F_{thrust} = 15{,}600N$, $g_{Moon} = 1.6m/s^2$
unknown: $a = ?$

Free-body diagram



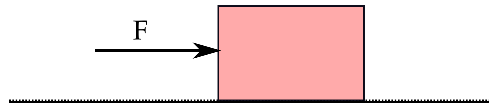

$$F_{thrust} = 15{,}600N$$
$$mg_{Moon} = 4550kg\left(1.6m/s^2\right) = 7280N$$

Newton's Second Law

$$\sum F_y = ma_y$$
$$F_{thrust} - mg_{Moon} = ma$$
$$\therefore a = \frac{F_{thrust} - mg_{Moon}}{m}$$
$$a = \frac{15{,}600N - 7280N}{4550kg}$$
$$a \approx 1.8m/s^2$$

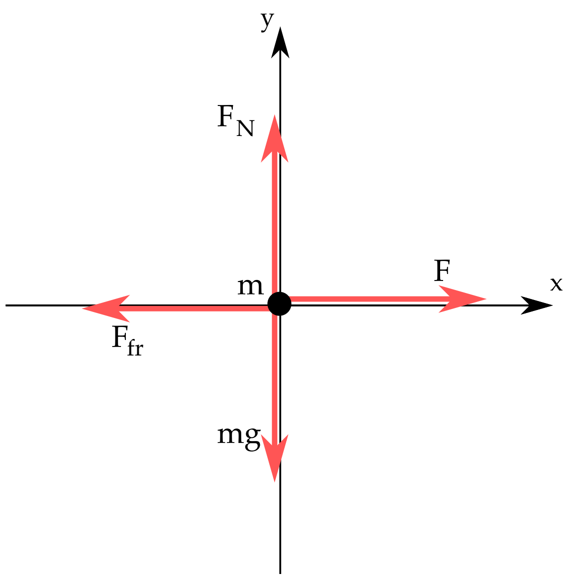
Note: this thrust is not enough to liftoff from the Earth.

Problem 3: A 100 kg wooden crate is at rest on a horizontal floor with a coefficient of static friction of 0.70 and coefficient of kinetic friction of 0.50. (a) What is the minimum horizontal force required to begin moving the block? (b) If the same force continues, how quickly will the block accelerate?




Solution:
part (a) given: $m = 100kg$, $\mu_s = 0.70$, $\theta = 0^\circ$, $a \geq 0$
part (a) unknown: $F = ?$

Free-body diagram




$$\begin{aligned} F &= ? \\ mg &= 980N, \\ F_N &= mg = 980N, \\ F_{fr} &= \mu_s F_N = \mu_s mg \end{aligned}$$

Newton's Second Law:

$$\begin{aligned} \sum F_x &= ma_x \\ F - F_{fr} &= m\cancel{a_x} \\ F - F_{fr} &\geq 0 \\ F &\geq F_{fr} \\ \therefore\ F &\geq \mu_s mg \\ F &\geq 0.70\left(100kg\right)\left(9.8m/s^2\right) \\ F &\geq 686N \end{aligned}$$

part (b) given: $m = 100kg$, $\mu_k = 0.50$, $\theta = 0^\circ$, $F = 686N$,

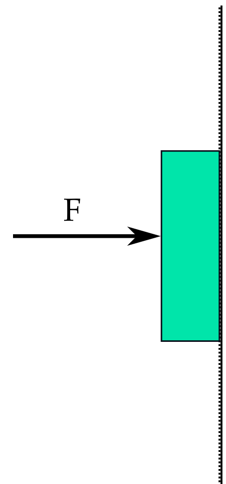
free-body diagram looks the same as part a

part (b) unknown: $a = ?$

$$\begin{aligned} F &= 686N \\ mg &= 980N, \\ F_N &= mg = 980N, \\ F_{fr} &= \mu_k F_N = 490N \end{aligned}$$

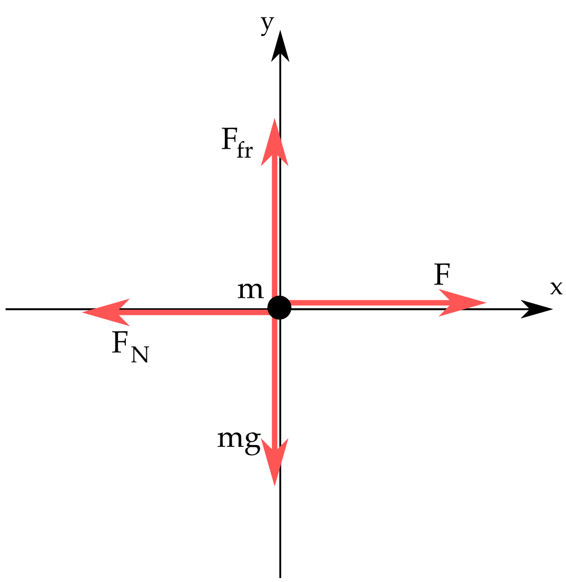
$$\begin{aligned} \sum F_x &= m a_x \\ F - F_{fr} &= m a_x \\ \therefore \; a_x &= \frac{F - \mu_k mg}{m} \\ a_x &= \frac{F - \mu_k mg}{m} \\ a_x &= \frac{686N - 490N}{100kg} \\ a_x &= 1.96m/s^2 \end{aligned}$$

Problem 4: A horizontal force presses a 4 kg textbook against a vertical wall. If the coefficient of friction between the wall and the book is 0.75, what is the minimum value of the force that would keep the book from falling?




Solution:
given: $m = 4kg$, $\mu = 0.75$, $a = 0$
unknown: $F = ?$

Free-body diagram: Normal forces are not always vertical. Remember the word "normal" in Physics comes form Math, not English. Normal means perpendicular. A normal force is perpendicular to a surface. Since a wall is vertical, the normal force is horizontal. Friction is always opposite the motion. Since it is to prevent the book from falling downward, the friction must be upward.



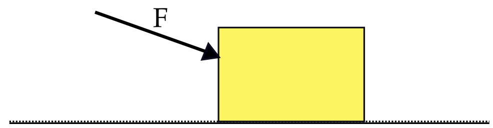

$$\begin{aligned} F &= ? \\ mg &= 39.2N, \\ F_N &= ?, \\ F_{fr} &= \mu F_N = ? \end{aligned}$$

$$\begin{aligned} \sum F_x &= ma_x \\ F - F_N &= m\cancel{a_x} \\ F - F_N &= 0 \\ F &= F_N \end{aligned}$$

Newton's Second Law:

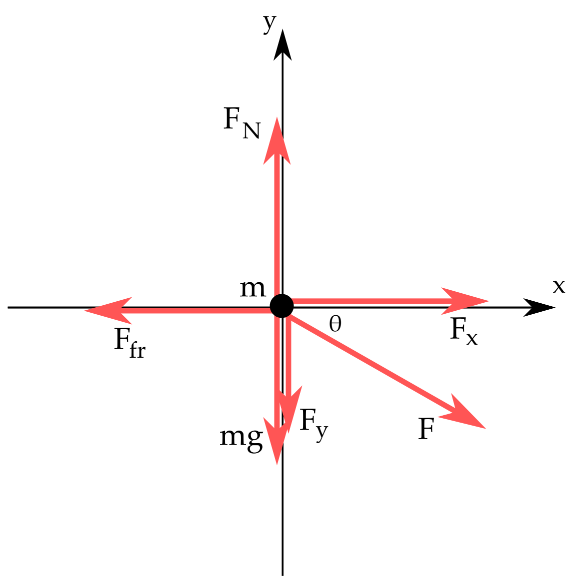
$$\begin{aligned} \sum F_y &= ma_y \\ F_{fr} - mg &= m\cancel{a_x} \\ \mu F_N - mg &= 0 \\ \mu F - mg &= 0 \\ \therefore \quad F &= \frac{mg}{\mu} \\ F &= \frac{4kg\left(9.8m/s^2\right)}{0.75} \\ F &\approx 52.3N \end{aligned}$$

Problem 5: A continuous 250 N push is applied 20° downward on a loaded 50 kg shopping cart. If there is a coefficient of friction of 0.40, how quickly will the cart accelerate down the supermarket aisle?




Solution:
given: $F = 250N$, $\theta = -20^{\circ}$, $m = 50kg$, $\mu = 0.40$
unknown: $a = ?$

Free-body diagram:

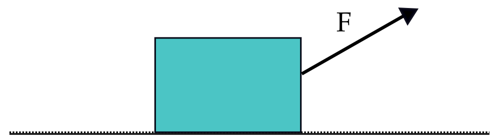



$$F_x = F\cos\theta \approx 234.9N,$$
$$F_x = F\sin\theta = 85.5N,$$
$$mg = 490N,$$
$$F_N = mg + F_y = 575.5N,$$
$$F_{fr} = \mu F_N = 230.2N$$

Newton's Second Law

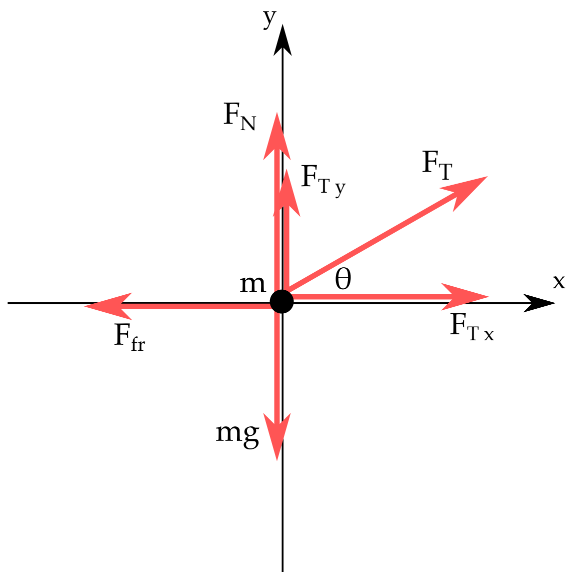
$$\sum F_x = ma_x$$
$$F_x - F_{fr} = ma_x$$
$$\therefore \; a_x = \frac{F_x - F_{fr}}{m}$$
$$a_x = \frac{234.9N - 230.2N}{50kg}$$
$$a_x \approx 0.094m/s^2$$

Problem 6: A tow-truck's cable pulls a 1800 kg car at constant speed, on a level road, against a 0.20 coefficient of friction. What is the cable's tension force? Assume the cable is 30° above the horizontal.




Solution:
given: $m = 1800kg$, $a = 0$, $\mu = 0.20$, $\theta = 30^{\circ}$
unknown: $F_T = ?$

Free-body diagram:



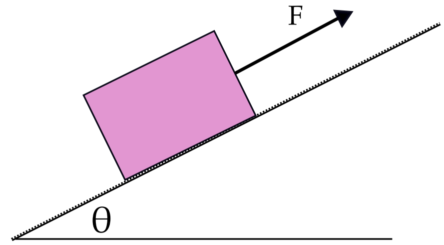

$$F_{Tx} = F_T \cos\theta,$$
$$F_{Ty} = F_T \sin\theta,$$
$$mg = 17{,}640N,$$
$$F_N = mg - F_T \sin\theta,$$
$$F_{fr} = \mu F_N = \mu\left(mg - F_T \sin\theta\right)$$

Newton's Second Law:

$$\sum F_x = m a_x$$
$$F_{Tx} - F_{fr} = m\cancel{a_x}$$
$$F_{Tx} = F_{fr}$$
$$F_T \cos\theta = \mu\left(mg - F_T \sin\theta\right)$$
$$= \mu mg - \mu F_T \sin\theta$$
$$F_T \cos\theta + \mu F_T \sin\theta = \mu mg$$

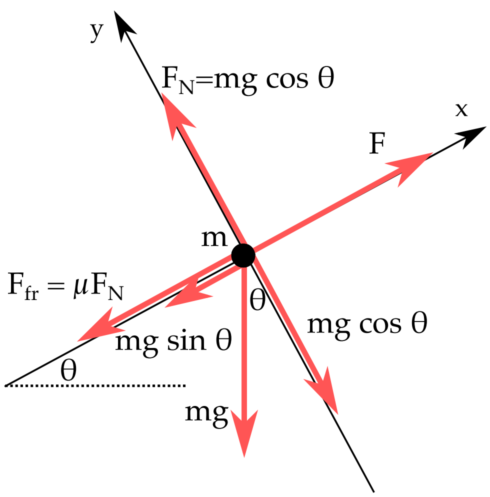
$$\therefore \quad F_T = \frac{\mu mg}{\cos\theta + \mu \sin\theta}$$
$$F_T = \frac{0.20\left(1800kg\right)\left(9.8m/s^2\right)}{\cos 30^{\circ} + 0.20 \sin 30^{\circ}}$$
$$F_T \approx 3650N$$

Problem 7: If you dragged a 50 kg box up a 30° inclined plane, with a 400 N force parallel to the incline, how quickly would the box accelerate? Assume that there is 0.35 coefficient of friction.




Solution:
given:
$m = 50kg$, $\theta = 30^{\circ}$, $F = 400N$, $\mu = 0.35$
unknown: $a = ?$

Free-body diagram: Since the normal force is perpendicular to the incline, the normal force is not vertical, and is not opposite to the weight.




$$
\begin{aligned}
F &= 400N \\
mg &= 490N, \\
mg \sin\theta &= 245N, \\
mg \cos\theta &\approx 424N, \\
F_N &= mg\cos\theta \approx 424N, \\
F_{fr} &\approx \mu F_N = 148N
\end{aligned}
$$

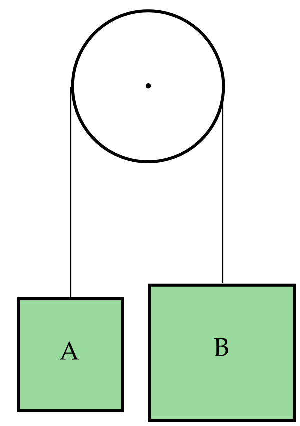
Newton's Second Law:

$$
\begin{aligned}
\sum F_x &= ma_x \\
F - mg\sin\theta - F_{fr} &= ma_x \\
\therefore\ a &= \frac{F - mg\sin\theta - F_{fr}}{m}
\end{aligned}
$$

$$a = \frac{424N - 245N - 148N}{50kg}$$

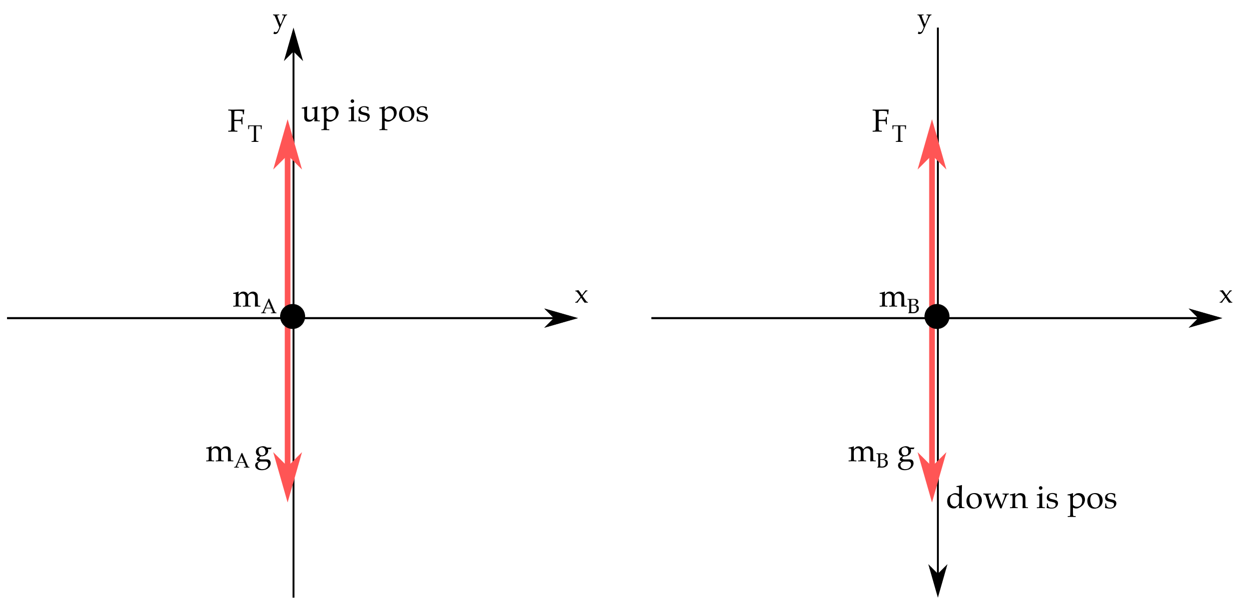
$$a \approx 0.62m/s^2$$

Problem 8: An Atwood machine consists of two masses connected by a cord, that is hung over a perfect pulley. If mass A is 10 kg and mass B is 15 kg, what is the acceleration of the system when released? What is the tension in the cord?




Solution:
given: $m_A = 10kg$, $m_B = 15kg$
unknown: $a = ?$, $F_T = ?$

Two masses means two free-body diagrams. Since mass B is larger, it is expected to fall, while mass A rises.




Two free-body diagrams means two Newton's Second Law calculations that can be then combined, because their accelerations are the same.

$$\sum F_y = m_A a_y \qquad\qquad \sum F_y = m_B a_y$$
$$F_T - m_A g = m_A a \qquad\qquad m_B g - F_T = m_B a$$

$$\begin{array}{r} \cancel{F}_T - m_A g = m_A a \\ +\left(m_B g - \cancel{F}_T = m_B a\right) \\ \hline m_B g - m_A g = m_A a + m_B a \end{array}$$

$$\therefore \; a = \frac{g\left(m_B - m_A\right)}{m_A + m_B}$$

$$a = \frac{9.8m/s^2\left(15kg - 10kg\right)}{10kg + 15kg}$$

$$a = 1.96m/s^2$$

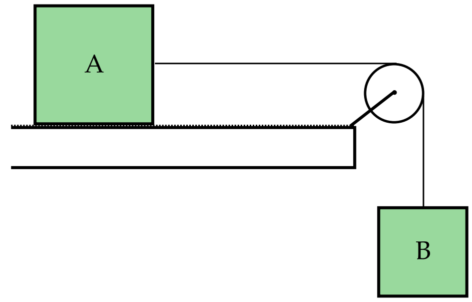
using mass A to find $F_T$:

$$F_T - m_A g = m_A a$$
$$F_T = m_A a + m_A g$$
$$\therefore \; F_T = m_A\left(a + g\right)$$
$$F_T = 10kg\left(9.8m/s^2 + 1.96m/s^2\right)$$
$$F_T \approx 118N$$

using mass B to find $F_T$:

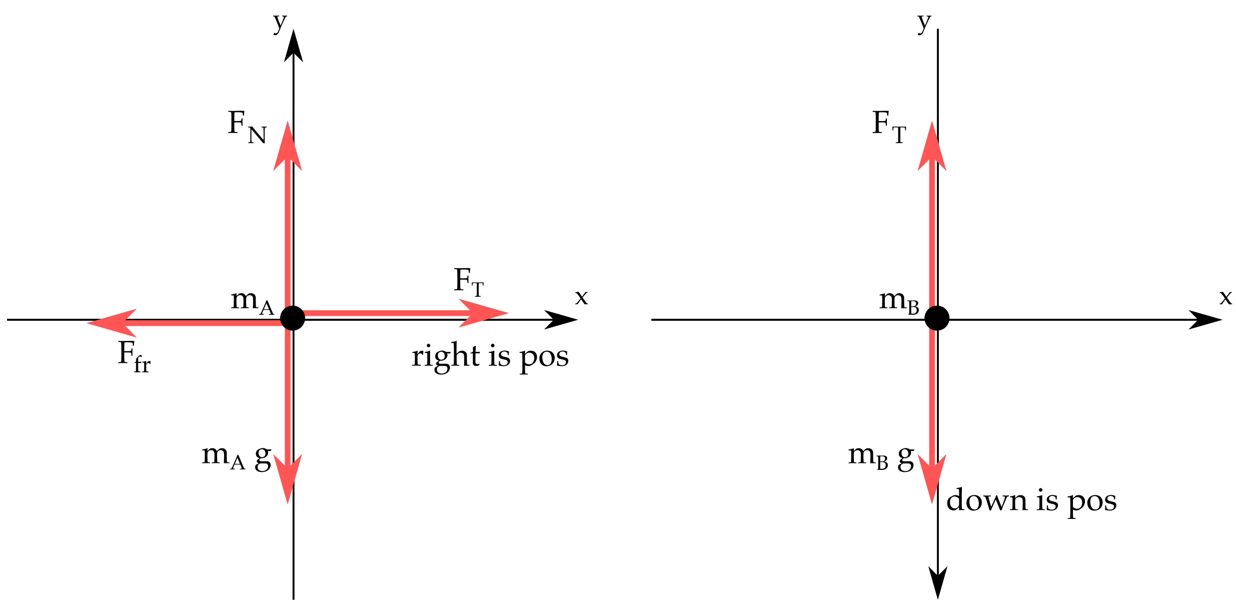
$$m_B g - F_T = m_B a$$
$$F_T = m_B g - m_B a$$
$$\therefore \; F_T = m_B\left(g - a\right)$$
$$F_T = 15kg\left(9.8m/s^2 - 1.96m/s^2\right)$$
$$F_T \approx 118N$$

Problem 9: Two masses are connected by a cord. One mass is on a table. The other mass is hung over a perfect pulley. If mass A is 15 kg and mass B is 12 kg, what is the minimum coefficient of static friction of the table that will keep system from moving when mass B is released?




Solution:
given: $m_A = 12kg$, $m_B = 15kg$, $a = 0$
unknown: $\mu_s = ?$

Two masses needs two free-body diagrams. Assume mass B has tendency to fall for positive/negative directions.




Two free-body diagrams means two Newton's Second Law calculations that can be then combined, because their accelerations are the same.

$$\sum F_x = m_A a_x \qquad \sum F_y = m_B a_y$$

$$F_T - F_{fr} = m_A \cancel{a} \qquad m_B g - F_T = m_B \cancel{a}$$

$$F_T - \mu_s m_A g = 0 \qquad m_B g - F_T = 0$$

$$\begin{array}{r} \cancel{F_T} - \mu_s m_A g = 0 \\ +\left(m_B g - \cancel{F_T} = 0\right) \\ \hline m_B g - \mu_s m_A g = 0 \end{array}$$

$$\therefore\ \mu_s = \frac{m_B}{m_A}$$

$$\mu_s = \frac{12kg}{15kg}$$

$$\mu_s = 0.80$$

# Part 5: Circular motion

**"The generalizations of science sweep in ever widening circles, and more aspiring flights through a limitless creation." T. H. Huxley (1825 – 1895)**

Circular (or centripetal) motion means turning around in a circle.

Turning is not the same as spinning "It's perfectly OK for you to turn your car around in a circle, but if your car starts spinning, you're about to crash."

Since velocity is a vector - a two part measurement - when we say that acceleration is the change of velocity, we can change either part.

Centripetal (or radial) acceleration is change of velocity DIRECTION; direction change is inward on radius line. Its magnitude is:

$$\left|a_c\right| = \frac{|v|^2}{r}$$

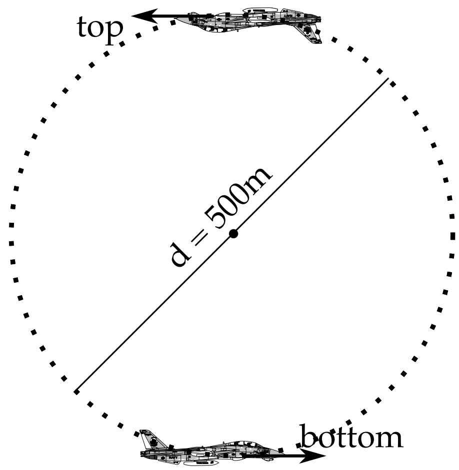
Linear (or tangential) acceleration is change of velocity MAGNITUDE; change is along a straight tangent line.

"We will not be changing both velocity magnitude and direction at the same time in this class. (Under normal circumstances.) Remember, you are taught in driver's ed NOT to step on the gas or brake AND turn the steering wheel at the same time."

Since we have a special version of acceleration, there is a centripetal force version of Newton's Second Law

$$\sum F_c = m\frac{v^2}{r}$$

## Part 5 Circular motion Problems



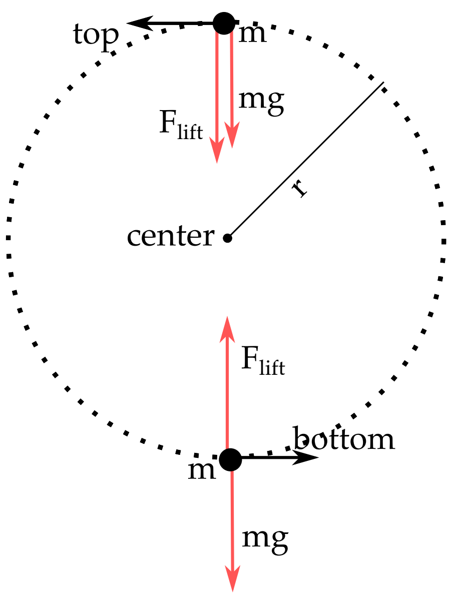

Problem 1: At an airshow, you watch a display where a 12,000 kg airplane flies a 500 m diameter vertical loop at a constant speed of 200 knots (103 m/s). What is the airplane's lift force at the top and the bottom of the loop?

Solution:
given:

$m = 12{,}000kg,\ r = \dfrac{d}{2} = 250m,\ v = 103m/s$

unknown: $F_{lift} = ?$

The top and bottom have different free-body diagrams. For circles, the x-y plane is inappropriate. For a circle, toward the center is positive, and directly away from the center is negative.




$$\sum F_{top} = m\frac{v^2}{r}$$

$$F_{lift} + mg = m\frac{v^2}{r}$$

$$\therefore\ F_{lift} = m\frac{v^2}{r} - mg$$

$$= 12{,}000kg\,\frac{(103m/s)^2}{250m} - 12{,}000kg\left(9.8m/s^2\right)$$

$$\approx 509{,}200N - 117{,}600N$$

$$F_{lift} \approx 391{,}600N$$

$$\sum F_{bottom} = m\frac{v^2}{r}$$

$$F_{lift} - mg = m\frac{v^2}{r}$$

$$F_{lift} = m\frac{v^2}{r} + mg$$

$$= 12{,}000kg\,\frac{(103m/s)^2}{250m} + 12{,}000kg\left(9.8m/s^2\right)$$

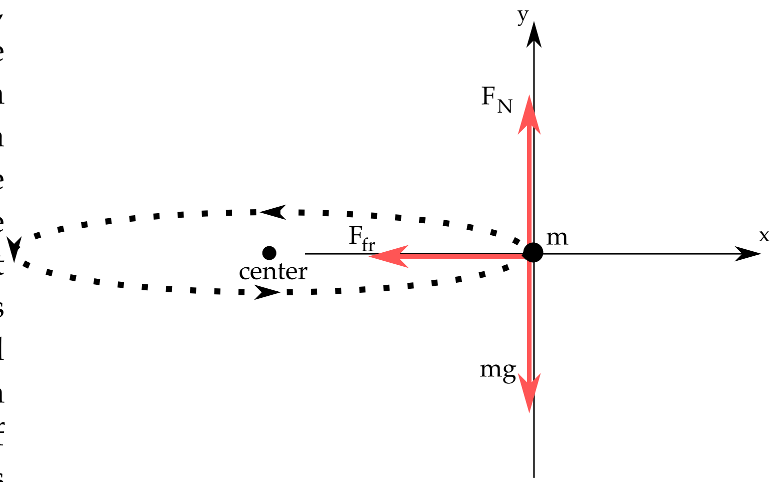
$$\approx 509{,}200 + 117{,}600N$$

$$F_{lift} \approx 626{,}800N$$

Problem 2: The Mazda Miata sports car is not very swift, but it is very nimble. If there is a coefficient of friction of 0.95 between a Miata's tires and asphalt, how fast can a Miata safely drive around a 100 m radius curve?

Solution:
given: $\mu = 0.95,\ r = 100m$
unknown: $v = ?$

Free-body diagram: Remember, "centripetal" means "toward the center." Centripetal acceleration is toward the center, because a constant velocity would increase radius and move away from the center. Since friction would resist this outward velocity, friction is toward the center in centripetal motion, not backwards as in linear acceleration. This is one of the few instances when friction is useful.

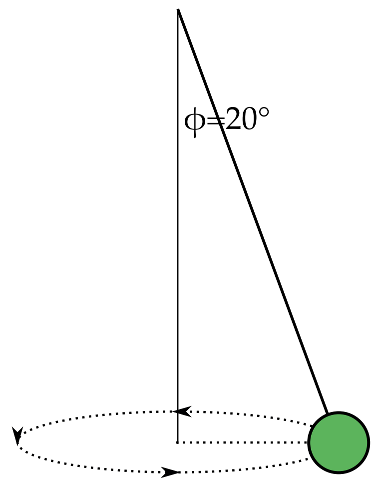



Newton's Second Law: horizontal circle

$$\sum F_x = m a_x$$
$$\sum F_c = m\frac{v^2}{r}$$
$$F_{fr} = m\frac{v^2}{r}$$
$$\mu \cancel{m} g = \cancel{m}\frac{v^2}{r}$$
$$\therefore\ v = \sqrt{\mu g r}$$
$$v = \sqrt{0.95\left(9.8m/s^2\right)(100m)}$$
$$v \approx 30.7m/s$$

Notice that the mass cancels out in the Math. This is somewhat common in pencil-and-paper Physics problems. All objects in this class have a mass, but that does not mean it affects what is being analyzed.

Problem 3: A regular pendulum swings back-and-forth in an arc. A conical pendulum swings around in a horizontal circle. If a conical pendulum swings around a circle with a 1 m diameter, so that the pendulum cord makes $\phi = 20°$ with the vertical, what is is the centripetal speed of the pendulum?

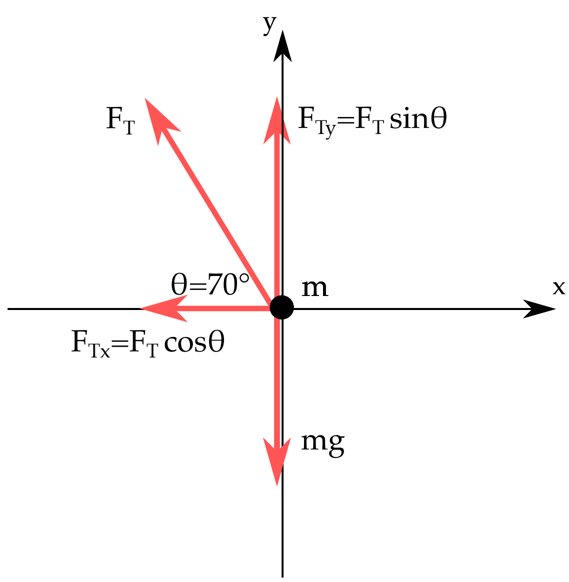



Solution: the 20° angle is not the mass' angle. The geometry says θ=70°.

given: $r = \dfrac{d}{2} = 0.5m$, $\phi = 20°$, $\theta = 70°$

unknown: $v = ?$

Free-body diagram:



Newton's Second Law: horizontal circle

$$\sum F_x = ma_x$$
$$\sum F_c = m\frac{v^2}{r}$$
$$F_{Tx} = m\frac{v^2}{r}$$
$$F_T\cos\theta = m\frac{v^2}{r}$$
$$\therefore\ F_T = \frac{mv^2}{r\cos\theta}$$

The centripetal velocity, mass and tension are all unknown. Check vertical forces to see if the situation simplifies.

Newton's Second Law: vertical forces

$$\sum F_y = m_B a_y$$
$$F_{Ty} - mg = m_B \cancel{a}$$
$$F_T\sin\theta - mg = 0$$
$$\therefore\ F_T = \frac{mg}{\sin\theta}$$

$$\therefore\ \frac{\cancel{m}v^2}{r\cos\theta} = \frac{\cancel{m}g}{\sin\theta}$$
$$v^2 = \frac{gr\cos\theta}{\sin\theta}$$
$$\therefore\ v = \sqrt{\frac{gr}{\tan\theta}}$$
$$v = \sqrt{\frac{9.8m/s^2\left(0.5m\right)}{\tan 70^\circ}}$$
$$v \approx 1.34m/s$$

# Part 6: Gravitation

**"One thing I have learned in a long life: that all our science, measured against reality, is primitive and childlike – and yet it is the most precious thing we have." Albert Einstein (1879 – 1955)**

Newton's Law of Universal Gravitation says that any two (or more) masses exert a gravitational force on each other, that tends to pull them together:

$$F_g \;=\; G\frac{m_1\ m_2}{r^2}$$

or in vector notation:

$$\overrightarrow{F_g} \;=\; G\frac{m_1\ m_2}{r^2}\ \hat{r}$$

G is the Universal Gravitational Constant: $G = 6.67 \times 10^{-11} N \cdot m^2/kg^2\ (m^3/s^2 \cdot kg)$

Since gravitational forces are forces, they can be added like any other force. This is called the superposition of forces.

$$\overrightarrow{F}_{1,\,net} = \sum_{i=2}^{n} \overrightarrow{F}_{1,\,i} = \overrightarrow{F}_{1,\,2} + \overrightarrow{F}_{1,\,3} + \ldots + \overrightarrow{F}_{1,\,n}$$

Since G is tiny, most masses are not big enough to yield a gravitational force large enough to be noticeable to you. The major exception is the entire Earth itself. Your weight is the gravitational force between you and the Earth.

$$m_{object}\ g_{planet} \;\equiv\; G\frac{m_{planet}\ m_{object}}{r^2_{planet}}$$

This means that gravitational acceleration is directly related to a planets' dimensions:

$$g_{planet} \;=\; G\frac{m_{planet}}{r^2_{planet}}$$

This also means a planet's gravitational acceleration if you are above the surface:

$$g_{altitude} \;=\; G\frac{m_{planet}}{r^2_{center}}$$

Remember, the r is the radial distance (direct line) measured from the center of a planet, not from the surface (the height).

---

## SATELLITE MOTION

Projectile motion, as discussed earlier, assumes the world is flat. The Earth is not flat. If a projectile is thrown less than a few thousand meters, we can pretend the Earth is flat. If a projectile is thrown thousands of kilometers, we must account for the spherical Earth curving down under the projectile. If a projectile is thrown correctly, it will continually "miss" hitting the Earth and end up circling the Earth. Satellite motion is a special type of circular motion, where the centripetal force is gravitational force.

$$F_c = F_g$$

$$\cancel{m_{satellite}} \frac{v_{satellite}^2}{r_{orbit}} = G \frac{m_{planet}\ \cancel{m_{satellite}}}{r_{orbit}^2}$$

therefore, satellite speed is:

$$v_{satellite} = \sqrt{G \frac{m_{planet}}{r_{orbit}}}$$

The velocity definition can then be used to determine satellite period - the time needed to complete one orbit:

$$T_{satellite} = \frac{2\pi r_{orbit}}{v_{satellite}}$$

# Part 6 Gravitation Problems

Problem 1: In lab, you measured the gravitational acceleration of the Earth as approximately 9.8 m/s². From New York City's 40° North latitude, and known 4450 km distance north of the equator, simple geometry can be used to determine the Earth's diameter as about 12,760 km. Use this information to determine the mass of the Earth in kg.

Solution:

given: $g_{Earth} = 9.8m/s^2,\ r_{Earth} = \frac{d_{Earth}}{2} = 6.38 \times 10^6 m,\ G = 6.67 \times 10^{-11} N \cdot m^2/kg^2$

unknown: $m_{Earth} = ?$

$$g_{Earth} = G\frac{m_{Earth}}{r^2_{Earth}}$$

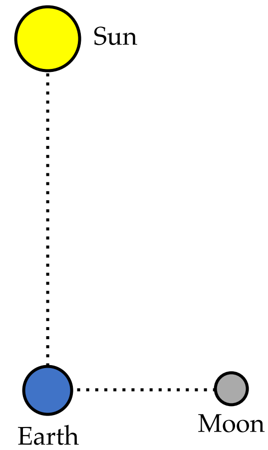
$$\therefore \quad m_{Earth} = \frac{g_{Earth}\ r^2_{Earth}}{G}$$

$$m_{Earth} = \frac{9.8m/s^2\left(6.38 \times 10^6 m\right)^2}{6.67 \times 10^{-11} N \cdot m^2/kg^2}$$

$$m_{Earth} \approx 5.98 \times 10^{24} kg$$

Problem 2: As the Earth orbits the Sun, and the Moon orbits the Earth, they will align at a right angle every two weeks. What is the net gravitational force (magnitude and direction) on the Earth, from the Sun and Moon, in this alignment?

The masses of the Earth, Moon and Sun are $5.98 \times 10^{24} kg$, $7.35 \times 10^{22} kg$, and $1.99 \times 10^{30} kg$, respectively. The Earth-Sun and Earth-Moon distances are $1.50 \times 10^{11} m$, and $3.84 \times 10^{8} m$. The Gravitational Constant is $6.67 \times 10^{-11} N \cdot m^2/kg^2$



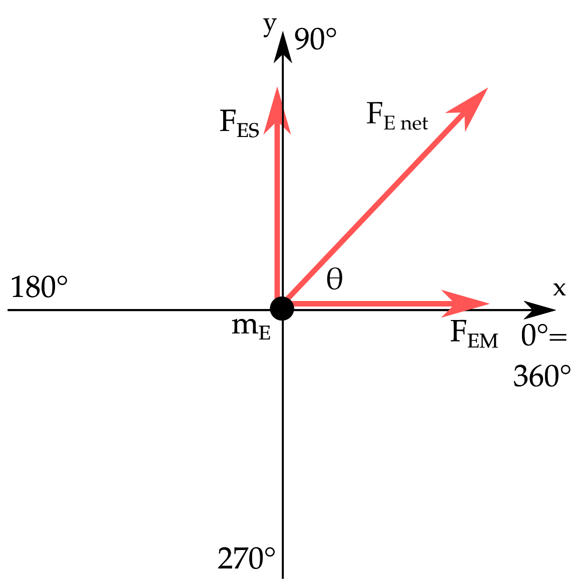

Solution:
given:

$$m_{Earth} = 5.98 \times 10^{24} kg,\ m_{Moon} = 7.35 \times 10^{22} kg,\ m_{Sun} = 1.99 \times 10^{30} kg,$$

$$r_{Earth-Sun} = 1.50 \times 10^{11} m,\ r_{Earth-Moon} = 3.84 \times 10^{8} m$$

unknown: $F_{E\ net} = ?$

Free-body diagram for gravitational forces:



Universal Gravitation:

$$F_{ES} = G\frac{m_E\, m_S}{r_{ES}^2}$$

$$F_{ES} = 6.67 \times 10^{-11} N \cdot m^2/kg^2 \frac{5.98 \times 10^{24} kg\left(1.99 \times 10^{30} kg\right)}{\left(1.50 \times 10^{11} m\right)^2}$$

$$F_{ES} \approx 3.53 \times 10^{22} N$$

$$F_{EM} = G\frac{m_E\, m_M}{r_{ES}^2}$$

$$F_{EM} = 6.67 \times 10^{-11} N \cdot m^2/kg^2 \frac{5.98 \times 10^{24} kg\left(7.35 \times 10^{22} kg\right)}{\left(3.84 \times 10^{8} m\right)^2}$$

$$F_{EM} \approx 1.99 \times 10^{20} N$$

$$F_{E\ net} = \sqrt{F_{ES}^2 + F_{EM}^2}$$

$$F_{E\ net} = \sqrt{\left(3.53 \times 10^{22} N\right)^2 + \left(1.99 \times 10^{20} N\right)^2}$$

$$F_{E\ net} \approx 3.53 \times 10^{22} N$$

$$\theta = \tan^{-1}\left(\frac{F_{ES}}{F_{EM}}\right)$$

$$\theta = \tan^{-1}\left(\frac{3.22 \times 10^{22} N}{1.99 \times 10^{20} N}\right)$$

$$\theta \approx 89.7^\circ$$

Problem 3: Popular accounts of the 1968 - 1972 Apollo voyages to the Moon often say that the spacecraft were rocketed all the way to the Moon. This is wrong. The Apollo spacecraft needed only to reach the gravitational force "equilibrium point" between the Earth and the Moon, after which the spacecraft would simply "fall" the rest of the way to the Moon. What is the equilibrium distance along a straight line between the Earth and the Moon? The masses of the Earth and Moon are $5.98 \times 10^{24} kg$ and $7.35 \times 10^{22} kg$, respectively. The Earth-Moon distance is $3.84 \times 10^{8} m$. The Gravitational Constant is $6.67 \times 10^{-11} N \cdot m^2/kg^2$

Solution:

given: $$\begin{gathered} m_{Earth} = 5.98 \times 10^{24} kg,\ m_{Moon} = 7.35 \times 10^{22} kg, \\ r_{Earth-eq} = x,\ r_{Moon-eq} = 3.84 \times 10^{8} m - x \end{gathered}$$

$$\begin{aligned} F_{Apollo-Earth} &= F_{Apollo-Moon} \\ \cancel{G}\frac{\cancel{m_{Apollo}}\, m_{Earth}}{r_{Earth-eq}^2} &= \cancel{G}\frac{\cancel{m_{Apollo}}\, m_{Moon}}{r_{Moon-eq}^2} \\ \frac{m_{Earth}}{r_{Earth-eq}^2} &= \frac{m_{Moon}}{r_{Moon-eq}^2} \end{aligned}$$

$$\begin{aligned} \frac{5.98 \times 10^{24}}{x^2} &= \frac{7.35 \times 10^{22}}{\left(3.84 \times 10^{8} - x\right)^2} \\ \frac{2.4454 \times 10^{12}}{x} &= \frac{2.7111 \times 10^{11}}{3.84 \times 10^{8} - x} \\ 9.3903 \times 10^{20} - 2.4454 \times 10^{12}\ x &= 2.7111 \times 10^{11}\ x \\ 9.3903 \times 10^{20} &= 2.7165 \times 10^{12}\ x \\ \therefore \quad 3.457 \times 10^{8}\ m &\approx x \equiv r_{Earth-eq} \\ \text{or} \quad r_{Moon-eq} &= 3.84 \times 10^{8} m - 3.457 \times 10^{8} m = 3.83 \times 10^{7}\ m \end{aligned}$$

Problem 4: The US Air Force's 27 Global Positioning System (GPS) satellites constantly transmit timing signals, that receivers use to fix locations on Earth to within inches, while orbiting at an altitude of 12,550 miles ($2.02\times10^{7}$ meters). What are the speed and period of the GPS satellites? The mass of the Earth is $5.98\times10^{24}$kg and the radius of the Earth is $6.38\times10^{6}$ m.

Solution:

given: $m_{Earth} = 5.98 \times 10^{24} kg,\ r_{orbit} = r_{earth} + h_{orbit} = 2.658 \times 10^{7} m$

unknown: $v_{GPS} = ?,\ T_{GPS} = ?$

$$v_{GPS} = \sqrt{G\frac{m_{Earth}}{r_{orbit}}}$$

$$v_{GPS} = \sqrt{6.67 \times 10^{-11} m^3/kg \cdot s^2 \ \frac{5.98 \times 10^{24} kg}{2.658 \times 10^{7} m}}$$

$$v_{GPS} \approx 3870 m/s \approx 8660 mph$$

$$T_{GPS} = \frac{2\pi r_{orbit}}{v_{GPS}}$$

$$T_{GPS} = \frac{2\pi\left(2.658 \times 10^{7} m\right)}{3870 m/s}$$

$$T_{GPS} \approx 43{,}200 s \approx 12 hr$$

Notice: although the speed is very fast, the period is still very long, because the circle is enormous: $2\pi r \approx 2(3.14)\left(2.658 \times 10^{7} m\right) \approx 1.67 \times 10^{8} m \approx 104{,}000\ miles$.

# Part 7: Work, Energy and Power

**“Mathematics, rightly viewed, possesses not only truth, but supreme beauty … such as only the greatest art can show.” Bertrand Russell (1872 – 1970)**

“Work” and “energy” are ordinary English words. Regular people use them in regular conversation. ‘I have to go to work, but I don’t have the energy to get up.’ That has absolutely nothing to do with what they mean in Physics. Do not mix up English and Physics.

Work is the “effort” to cause anything that does not occur naturally; energy is used to do work; doing work transfers energy.

---

## DEFINITION OF WORK

Work is strictly defined in Physics by mathematical equation:

$$W = F_{\|}d = F\,d\cos\phi_{Fd}$$

Think back to earlier, and imagine a horizontal force applied to an object on the floor. If the friction isn’t too large, the mass would move a distance. In Physics, we say that work is done when a force is applied to cause an object to displace. Be careful, there must be both an external force AND a displacement. If there is no motion, even if there is a force, there is no work being done. If an object moves without there being an external force acting on it - if you drop something, for example - there is no work being done either.

In more complex situations, the mathematical definition of work can be a vector dot product or an integral (PHY 1300 only)

$$W = \vec{F} \bullet \vec{d} = \left|F\right| \left|d\right| \cos\phi_{Fd}$$

$$W = \int F(x)\,dx$$

---

## WORK TRANSFERS ENERGY

“What have you done, if you have done work?” The First Law of Thermodynamics says energy can neither be created nor destroyed, only transformed from one kind into another, or transferred from one place or thing to another. When work is done on anything, the work transfers energy to it.

Work can make an object change velocity. The energy of motion is kinetic energy. The official abbreviation for kinetic energy is K. However, most people like to write KE.

$$K \equiv KE = \frac{1}{2}mv^2$$

Notice that kinetic energy is not proportional to velocity, but rather velocity squared. This is one reason why almost any car you can buy will reach 100 mph, but only a handful will reach 200 mph: to double the maximum speed, the engine needs to be four times as powerful. (Even more when air resistance is accounted for.)

Work can lift an object against gravity. The energy of height is gravitational potential energy. The official abbreviation for potential energy is U. However, most people like to write PE.

$$U_{grav} \equiv PE_{grav} = mgh$$

Note: work and energy are scalars, not vectors - direction is not important. However, positive and negative values are still important to indicate a gain or loss of work and energy.

The word "potential" in gravitational potential energy does not refer to height; it refers to "saved" or "stored" energy. Work is done in lifting an object to a height against gravity, and therefore puts in energy "into" the object. The energy is "stored" because it can be returned as motion and kinetic energy by dropping the object.

There are other "stored" potential energies. The most important is elastic potential energy - energy is stored in a spring or other elastic "stretchy" object, by applying a force to compress or stretch it. The "stretchiness" of an elastic object is defined through Hooke's Law:

$$F_{elastic} = -kx$$

where k is the "elastic constant" - the stiffness of the spring.

Work can compress or stretch a spring. The formula for elastic potential energy is:

$$U_{elastic} \equiv PE_{elastic} = \frac{1}{2}kx^2$$

The elastic potential energy of springs used to be important in powering many machines. The most famous were mechanical wind-up watches. Battery-powered electronics have replaced spring-powered machines in most modern situations.

## THE WORK-ENERGY THEOREM

The work-energy theorem is a mathematical statement of the work-energy relationship. There are different ways to write the theorem.

$$W_{net} \;=\; \Delta K \;\equiv\; \Delta KE$$

$$\sum W_{nc} \;=\; \Delta K \;+\; \Delta U \;\equiv\; \Delta KE \;+\; \Delta PE$$

$$W_{external} \;+\; KE_i \;+\; PE_i \;=\; KE_f \;+\; PE_f \;+\; W_{fr}$$

It is called a theorem, because it can be derived from Newton's Second Law. You are not responsible for the mathematical proof.

We need you to recognize that Newton's Second Law force and acceleration calculations are vector calculations; while work-energy theorem calculations are scalar calculations. They are Physically equivalent - they give the same answers. Many work-energy problems will look familiar, because they are. We want you to see they give the same answers. In real life, the technique used depends on which is easier. Usually, but not always, the work-energy theorem will be easier, because scalar math is simpler.

## CONSERVATION OF ENERGY

In science, there are fundamental concepts called the conservation laws. In science, "conservation" means "does not change." We having been studying "change" in earlier topics: velocity is change of position, acceleration is change of velocity; force causes acceleration, so force causes change. etc.

However, if everything is always changing, we don't know anything for sure - what is true today, changes overnight and is longer true tomorrow. Therefore, science also searches for the things do not change; that are certain. These are the conservation laws. I always say that the conservation laws are "the things that are eternal."

There are several different conservation laws. If you have taken Chemistry, you will already know conservation of mass - the mass of the reactants and products in a chemical reaction are equal. The first one in Physics is conservation of energy - the total energy will not change by itself - the total initial energy equals the total final energy.

$$KE_i \;+\; PE_i \;=\; KE_f \;+\; PE_f$$

Be careful, conservation of energy only applies in specific situations - work adds energy into a system, friction "steals" energy and dissipates it as heat. We say external and friction forces are nonconservative forces. Since there is always some friction in real life, conservation of energy is an approximation in real life.

In this class, roller coasters, pendulums, springs and projectile motion normally obey conservation of energy.

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## POWER

Power is rate of work produced or energy used. It is mathematically defined as:

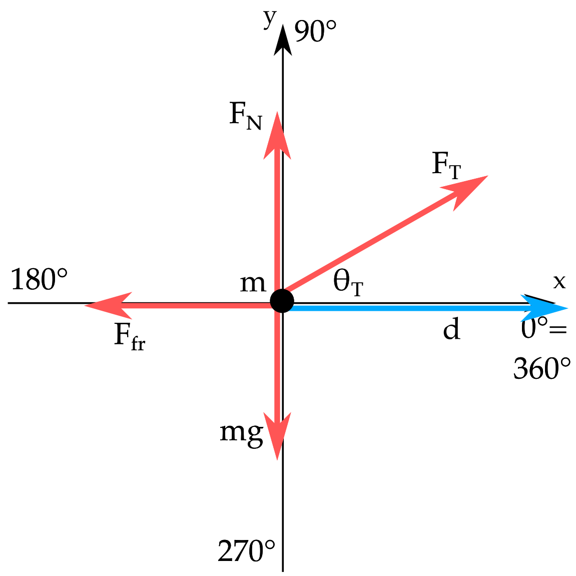
$$P = \frac{W}{t}$$

"Power" is also an ordinary English word, used in regular conversation, that also has absolutely nothing to do with what they mean in Physics. Do not mix up up English and Physics.

# Part 7 Work, Energy and Power Problems

Problem 1: A tow-truck's cable pulls a 1800 kg car at constant speed, a distance of 12 km on a level road, against a 0.15 coefficient of friction. How much work is done by the tension, gravitational, normal and friction forces on the car? What is the net work done on the car? Assume the cable is 30° above the horizontal.

Solution:
given: $m = 1800kg$, $a = 0$, $d = 12{,}000m$, $\mu = 0.15$, $\theta_T = 30^{\circ}$
from free-body diagram and Newton's Second Law:




$F_g \equiv mg = 17{,}640N$, $F_N = mg - F_T \sin\theta = 8820N$,

$F_{fr} \equiv \mu F_N = 2646N$, $F_T = \dfrac{F_{fr}}{\cos\theta} \approx 2646N$,

unknown: $W_T = ?$, $W_g = ?$, $W_N = ?$, $W_{fr} = ?$

$$W = F\,d\cos\phi_{Fd}$$
$$W_T = F_T\,d\cos\phi_{F_T d}$$
$$W_T = 2626N\,(12{,}000m)\cos 30^{\circ}$$
$$W_T \approx 2.75\times 10^{7}J$$

$$W = F\,d\cos\phi_{Fd}$$
$$W_g = mg\,d\cos\phi_{mg\,d}$$
$$W_g = 17{,}640N\,(12{,}000m)\cos 90^{\circ}$$
$$W_g = 0$$

$$W = F\,d\cos\phi_{Fd}$$
$$W_N = F_N\,d\cos\phi_{F_N d}$$
$$W_N = 17{,}640N\,(12{,}000m)\cos 90^{\circ}$$
$$W_N = 0$$

$$W = F\,d\cos\phi_{Fd}$$
$$W_{fr} = F_{fr}\,d\cos\phi_{F_{fr} d}$$
$$W_{fr} = 2646N\,\left(150m\right)\cos 180^{\circ}$$
$$W_{fr} = -2.75\times 10^{7}J$$

Notes: Joules values are very large for real-life situations, because one Joule is a very small amount of work or energy. The work done by normal forces is expected to be zero, because normal forces are usually perpendicular to the displacement. The work done by gravitational force in this problem is zero, because gravitational forces are downward, but the displacement in this problem is horizontal. The work done by friction forces is expected to be negative, because friction forces are usually opposite in direction to the displacement.

$$\begin{aligned} W_{net} &= W_T + W_g + W_N + W_{fr} \\ &= 2.75\times 10^{7}J + 0 + 0 + \left(-2.75\times 10^{7}\right) \\ W_{net} &= 0 \end{aligned}$$

Note: the net work is zero, because the speed is constant.

Problem 2: A tow-truck's cable pulls a 1800 kg car with a 5000 N tension force, 150 m up along a 10° ramp, against a 2000 N friction force. How much work is done by the tension, gravitational, normal and friction forces on the car? What is the net work done on the car? Assume the cable is parallel to the incline.

Solution:
given: $m = 1800kg,\ d = 150m,\ \theta_{incline} = 10^{\circ}$
from free-body diagram and Newton's Second Law:
$F_T = 5000N,\ F_g \equiv mg = 17{,}640N,\ F_N = mg\cos\theta \approx 17{,}372N,\ F_{fr} = 2000N$

unknown: $W_T = ?,\ W_g = ?,\ W_N = ?,\ W_{fr} = ?$

$$\begin{aligned} W &= F\,d\cos\phi_{Fd} \\ W_T &= F_T\,d\cos\phi_{F_T\,d} \\ W_T &= 5000N\,(150m)\cos 0^{\circ} \\ W_T &= 750{,}000J \end{aligned}$$

$$\begin{aligned} W &= F\,d\cos\phi_{Fd} \\ W_g &= mg\,d\cos\phi_{mg\,d} \\ W_g &= 17{,}640N\,(150m)\cos 100^{\circ} \\ W_g &\approx -459{,}500J \end{aligned}$$

$$\begin{aligned} W &= F\,d\cos\phi_{Fd} \\ W_N &= F_N\,d\cos\phi_{F_N\,d} \\ W_N &= 17{,}372N\,(150m)\cos 90^{\circ} \\ W_N &= 0 \end{aligned}$$

$$\begin{aligned} W &= F\,d\cos\phi_{Fd} \\ W_{fr} &= F_{fr}\,d\cos\phi_{F_{fr}\,d} \\ W_{fr} &= 2000N\,(150m)\cos 180^{\circ} \\ W_{fr} &= -300{,}000J \end{aligned}$$

Note: The work done by gravitational force in this problem is negative, because gravitational forces are downward, but the displacement in this problem is slightly upward.

$$\begin{aligned} W_{net} &= W_T + W_g + W_N + W_{fr} \\ &\approx 750{,}000J + (-459{,}500J) + 0 + (-300{,}000J) \\ W_{net} &\approx -9500J \end{aligned}$$

Note: the negative net work means the car is slowing down.

Problem 3: In a car accident investigation, it is important to determine a car's initial speed. (Speeding is evidence of negligent or even criminal behavior.) If a car skids for 56 m before stopping during a crash on a level road, against a coefficient of friction of 0.75, what is the car's initial speed?

Solution:
given: $d = 56m,\ v_f = 0,\ \mu = 0.75,\ h_0 = h_f = 0$

from free-body diagram and Newton's Second Law:
$F_{external} = 0,\ F_g \equiv mg,\ F_N = mg,\ F_{fr} = \mu mg$

unknown: $v_0 = ?$

Work-energy theorem:

$$W_{ext} + KE_i + PE_i = KE_f + PE_f + W_{fr}$$

$$\cancel{F_{ext}}d + \frac{1}{2}mv_0^2 + mg\cancel{h_0} = \frac{1}{2}m\cancel{v_f}^2 + mg\cancel{h_f} + F_{fr}d$$

$$\frac{1}{2}\cancel{m}v_0^2 = \mu\cancel{m}gd$$

$$\therefore\ v_0 = \sqrt{2\mu gd}$$

$$v_0 \approx 28.7m/s$$

Note: Since the work-energy theorem has six terms, you should cancel any unnecessary terms (because they equal zero) immediately. The theorem may simplify a lot.

This accident analysis technique is less necessary today, because today's cars have a built-in data recorder.

Problem 4: A tow-truck's cable pulls a 1800 kg car with a 5000 N tension force, 150 m up along a 10° ramp, against a 2000 N friction force. If the tow begins at 10 m/s at the bottom of the ramp, how fast will the car be moving when it reaches the top of the ramp? Assume the cable is parallel to the incline.

Solution:
given: $m = 1800kg$, $d = 150m$, $\theta_{incline} = 10^{\circ}$, $v_0 = 10m/s$, $h_0 = 0$

from free-body diagram and Newton's Second Law:
$F_T = 5000N$, $F_g \equiv mg = 17{,}640N$, $F_N = mg\cos\theta \approx 17{,}372N$, $F_{fr} = 2000N$

from the right triangle: $h_f = d\sin\theta = 150m\sin 10^{\circ} \approx 26.0m$

unknown: $v_f = ?$

Work-energy theorem:

$$W_{ext} + KE_i + PE_i = KE_f + PE_f + W_{fr}$$

$$Fd + \frac{1}{2}mv_0^2 + mg\cancel{h_0} = \frac{1}{2}mv_f^2 + mgh_f + F_{fr}d$$

$$5000N\left(150m\right) + \frac{1}{2}\left(1800kg\right)\left(10m/s\right)^2 =$$

$$\frac{1}{2}\left(1800kg\right)v_f^2 + 1800kg\left(9.8m/s^2\right)\left(26.0m\right) + 2000N\left(150m\right)$$

$$750{,}000J + 90{,}000J = 900kg\ \ v_f^2 + 458{,}640J + 300{,}000J$$

$$840{,}000J = 900kg\ \ v_f^2 + 758{,}640J$$

$$\therefore\ v_f = \sqrt{\frac{840{,}000J - 758{,}640J}{900kg}}$$

$$v_f \approx 9.5m/s$$

Note: cancelling zero value terms does not simplify the theorem much in this problem. I plugged in the data and simplified the numbers, before manipulating the expression. "Arithmetic is easier than algebra."

Problem 5: The first hill of the Cyclone roller coaster at Coney Island climbs up 85 ft (26 m) before dropping to ground level, and then rises again to a second 70 ft (21 m) hill. If a roller coaster car is released at the top of the first hill at 5 m/s, what would be the speed of the car when it reaches the bottom of the hill? What would be the speed of the car when it reaches the top of the second hill? Ignore friction.

Solution:
given: $v_{hill\,1} = 5m/s$, $h_{hill\,1} = 26m$, $h_{hill\,2} = 21m$
unknown: $v_{valley} = ?$, $v_{hill\,2} = ?$

$$\begin{aligned} hill_1 &= valley \\ KE_i + PE_i &= KE_f + \cancel{PE_f} \\ \frac{1}{2}\cancel{m}v_i^2 + \cancel{m}gh_i &= \frac{1}{2}\cancel{m}v_f^2 \\ \therefore \quad v_f &= \sqrt{v_i^2 + 2gh_i} \\ v_f &= \sqrt{\left(5m/s\right)^2 + 2\left(9.8m/s^2\right)\left(26m\right)} \\ v_f &\approx 23.1m/s \end{aligned}$$

$$\begin{aligned} hill_1 &= hill_2 \\ KE_i + PE_i &= KE_f + PE_f \\ \frac{1}{2}\cancel{m}v_i^2 + \cancel{m}gh_i &= \frac{1}{2}\cancel{m}v_f^2 + \cancel{m}gh_f \\ \therefore \quad v_f &= \sqrt{v_i^2 + 2gh_i - 2gh_f} \\ v_f &= \sqrt{\left(5m/s\right)^2 + 2\left(9.8m/s^2\right)\left(26m\right) - 2\left(9.8m/s^2\right)(21m)} \\ v_f &\approx 11.1m/s \end{aligned}$$

Problem 6: Coney Island's new Thunderbolt roller coaster vertically drops 110 ft (33.5 m) before entering a loop. If a roller coaster car begins at rest at the top of the drop, what is the maximum height of the loop, so that the car will safely go around the top of the loop? Assume the loop is circular (it's not) and ignore friction.

Solution:
given: $v_{drop} = 0$, $h_{drop} = 33.5m$, $h_{loop} = 2r_{loop}$
unknown: $v_{loop} = ?$, $h_{loop} = ?$

$$\begin{aligned}
\sum F_c &= m\frac{v^2}{r} \\
\cancel{F_N} + mg &= m\frac{v_{top}^2}{r} \\
\cancel{m}g &\le \cancel{m}\frac{v_{top}^2}{r} \\
\therefore \quad v_{loop} &\ge \sqrt{gr_{loop}}
\end{aligned}$$

$$\begin{aligned}
drop &= loop \\
\cancel{KE_i} + PE_i &= KE_f + PE_f \\
\cancel{m}gh_{drop} &= \tfrac{1}{2}\cancel{m}v_{loop}^2 + \cancel{m}gh_{loop} \\
\cancel{g}h_{drop} &= \frac{\cancel{g}r_{loop}}{2} + \cancel{g}h_{loop} \\
h_{drop} &= \frac{h_{loop}}{4} + h_{loop} \\
h_{drop} &= \frac{5h_{loop}}{4} \\
\therefore \quad h_{loop} &= \frac{4h_{drop}}{5} \\
h_{loop} &= \frac{4\left(33.5m\right)}{5} \\
h_{loop} &= 26.8m \approx 87.9ft
\end{aligned}$$

By the way, the loop is really about 60 ft (18.3 m) tall. Some mechanical engineers really do specialize in designing roller coasters.

Problem 7: Before the invention of gunpowder cannon, the most powerful weapon in the world was the trebuchet, a sling, lever and counterweight machine that could smash castle walls by hurling quarter-ton boulders the length of three football fields. Trebuchets use a falling counterweight to throw the projectile. If a 2000 kg counterweight falls 10 m from rest to a stop, to throw a 100 kg projectile that rises 20 m before release, how fast can the trebuchet throw the projectile.

Solution:

given: $$\begin{array}{l} m_{weight} = 2000kg,\ v_{0\,weight} = 0,\ v_{f\,weight} = 0,\ h_{i\,weight} = 10m,\ h_{f\,weight} = 0 \\ m_{proj} = 100kg,\ v_{0\,proj} = 0,\ h_{i\,proj} = 0,\ h_{f\,proj} = 20m \end{array}$$

unknown: $v_{f\,proj} = ?$

$$\begin{aligned} KE_i \ + \ PE_i \ &= \ KE_f \ + \ PE_f \\ \cancel{KE_{weight}} + PE_{weight} + \cancel{KE_{proj}} + \cancel{PE_{proj}} &= \cancel{KE_{weight}} + \cancel{PE_{weight}} + KE_{proj} + PE_{proj} \\ (mgh)_{weight} &= \left(\tfrac{1}{2}mv^2\right)_{proj} + (mgh)_{proj} \\ 2000kg\left(9.8m/s^2\right)(10m) &= \tfrac{1}{2}\left(100kg\right)v_{proj}^2 + 100kg\left(9.8m/s^2\right)(20m) \\ 196{,}000 &= 50v^2 + 19{,}600 \\ \therefore \quad v_{proj} &\approx 59.4m/s \end{aligned}$$

By the way, the word “engineer” originally meant a person who could design and build such “military engines.” However, engineers originally did their work through trial-and-error, not theoretical calculations.

Problem 8: A toy gun that contains a spring with an elastic constant of 54 N/m is used to shoot a 15 gram dart straight up into the air. If the gun is cocked by compressing the spring by 18 cm, how high will the dart shoot up, above the uncocked position? Ignore air resistance.

Solution:

given:
$$k = 54N/m,\ m = 0.015kg,$$
$$\text{uncocked } h = 0,\ h_0 = -0.18m,$$
$$x_0 = 0.18m,\ x_f = 0,$$
$$v_0 = 0,\ v_{top} = 0$$

unknown: $h_{top} = ?$

$$\cancel{KE_i} + PE_i = KE_f + PE_f$$
$$PE_{grav} + PE_{elastic} = PE_{grav} + \cancel{PE_{elastic}}$$
$$mgh_0 + \frac{1}{2}kx_0^2 = mgh_f$$
$$\therefore\ h_f = \frac{mgh_0 + \frac{1}{2}kx_0^2}{mg}$$
$$h_f = \frac{0.015kg\left(9.8m/s^2\right)(-0.18m) + \frac{1}{2}\left(54N/m\right)(0.18m)^2}{0.015kg\left(9.8m/s^2\right)}$$
$$h_f \approx 5.77m$$

Problem 9: The 3380 lb (1535 kg) Ford GT supercar can accelerate from rest to 150 mph (67.1 m/s) in 14.5 s. How powerful is the GT's engine?

Solution:
given: $m = 1535kg$, $v = 67.1m/s$, $t = 14.5s$
unknown: P = ?

$$P = \frac{W}{t}$$

$$P = \frac{KE + \cancel{PE}}{t}$$

$$\therefore \quad P = \frac{\frac{1}{2}mv^2}{t}$$

$$P = \frac{\frac{1}{2}\left(1535kg\right)\left(67.1m/s\right)^2}{14.5s}$$

$$P \approx 2.38 \times 10^5 kg \cdot m^2/s^3$$

$$P \approx 2.38 \times 10^5 Watt \approx 320\, horsepower$$

Note: since the GT's engine is rated at 647 hp, the losses to air resistance, tire slippage, etc., are over 50%.

Problem 10: The Saturn V rocket that launched the Apollo spacecraft to the Moon was the most powerful engine ever built. It used a technique called staging to conserve fuel. As fuel tanks emptied, their dead weight, and the rockets they fed, would be released, and fresh tanks and rockets would be lit to continue the voyage. The Saturn V first stage is ejected 2 ½ minutes into the flight after consuming 4.3 million pounds of fuel, at a height 41 miles (66,000 m) and a speed of 6200 mph (2770 m/s). What is the average power of a Saturn V rocket? Since the launch weight was 6.2 million pounds, assume a 4.05 million lb (1,837,000 kg) average mass. Ignore air resistance, and the weakening of gravity at very high altitudes.

$$P = \frac{W}{t}$$

$$P = \frac{KE + PE}{t}$$

$$P = \frac{\frac{1}{2}mv^2 + mgh}{t}$$

$$P = \frac{\frac{1}{2}\left(1{,}837{,}000kg\right)(2770m/s)^2 + 1{,}837{,}000kg\left(9.8m/s^2\right)\left(66{,}000m\right)}{150s}$$

$$P \approx \frac{7.048 \times 10^{12}J + 1.188 \times 10^{12}J}{150s}$$

$$P \approx 5.49 \times 10^{10}W \approx 7.4 \times 10^7 hp$$

Note, this is more power than a million cars. Note also, the known power is twice this because of air resistance, etc.

Problem 11: During World War 2, Grumman Aircraft of Bethpage, Long Island built over 12,000 F6F Hellcat fighters for the US Navy. The Hellcat used a 2000 horsepower (1490 kiloWatt) Pratt & Whitney R-2800 Double Wasp engine (made in East Hartford, CT). How quickly can a 12,200 lb (5530 kg) Hellcat take off from rest, on the ground, and reach a patrol altitude of 20,000 ft (6100 m) and a cruising speed of 250 mph (112 m/s)? Assume 50% overall efficiency.

Solution:

given: $P_{50\%} = 0.50\left(1.49 \times 10^6\, Watt\right),\ m = 5530kg,$
$v_0 = 0,\ h_0 = 0,\ v_f = 112m/s,\ h_f = 6100m$

unknown: t = ?

$$P = \frac{W}{t}$$

$$P = \frac{KE + PE}{t}$$

$$\therefore \quad t = \frac{\frac{1}{2}mv^2 + mgh}{P}$$

$$t_{50\%} = \frac{\frac{1}{2}\left(5530\right)\left(112\right)^2 + 5530\left(9.8\right)\left(6100\right)}{0.50\left(1.49 \times 10^6\right)}$$

$$t_{50\%} \approx 490\,s \approx 8.2\,min$$

# Part 8: Linear momentum

**"It is not so much I have confidence that scientists are right, but that I have confidence that nonscientists are wrong." Isaac Asimov (1920 – 1992)**

Momentum is another word that has everyday English meanings that have nothing to do with Physics. In fact, there is no real way to describe, in words, what momentum means in Physics. The definition is purely mathematical. The definition of linear momentum is:

$$\vec{p} = m\vec{v}$$

Notice, since velocity is a vector, momentum is also a vector. In other words, pay attention to the direction and positive and negative signs. Also pay attention to the difference between horizontal and negative.

Newton's Second Law was originally written by Newton in terms of momentum:

$$\sum \vec{F} = \frac{\Delta \vec{p}}{\Delta t}$$

Any force problem can be be done in terms of momentum. However, we will not. Momentum is normally applied to collisions - objects crashing into each other. Be aware: an explosion is the opposite of a collision, and can also be analyzed by momentum.

We usually rewrite Newton's Second Law as the impulse-momentum theorem

$$J = F\ \Delta t = \Delta\,(mv)$$

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## CONSERVATION OF MOMENTUM

The second conservation law discussed in Physics is conservation of momentum. If you don't interfere with a collision, the total momentum before and after the collision is the same.

$$p_i = p_f$$

There are two basic types of collisions - elastic vs. inelastic collisions - so the conservation of momentum is a little different between the two.

For an inelastic collision (hit & stick - the final mass is one total mass):

$$m_1\overrightarrow{v_{1i}} \ + \ m_2\overrightarrow{v_{2i}} \ = \ \left(m_1 \ + \ m_2\right) \ \overrightarrow{v_f}$$

Be aware: there is always a decrease of kinetic energy in an inelastic collision. Theoretically, it "lost" as heat; in reality, the energy would also do the work of damaging the objects.

For an elastic collision (hit & bounce - the final masses remain separate):

$$m_1\overrightarrow{v_{1i}} \ + \ m_2\overrightarrow{v_{2i}} \ = \ m_1\overrightarrow{v_{1f}} \ + \ m_2\overrightarrow{v_{2f}}$$

Be aware: the total kinetic energy is also conserved in a perfect elastic collision.

$$\tfrac{1}{2}m_1v_{1i}^2 \ + \ \tfrac{1}{2}m_2v_{2i}^2 = \ \tfrac{1}{2}m_1v_{1f}^2 \ + \ \tfrac{1}{2}m_2v_{2f}^2$$

This is normally simplified to:

$$v_{2f} \ - \ v_{1f} \ = \ v_{1i} \ + \ v_{2i}$$

## Part 8 Linear momentum Problems

Problem 1: The most powerful aircraft cannon ever flown is carried by the US Air Force's A-10, called the Warthog. The Warthog uses its 30mm Avenger cannon to kill tanks. The Avenger cannon is reputed to be so powerful that firing it produces enough recoil to noticeably slow a 'Hog in flight. Since the Avenger fires 13.9 oz (395 g) bullets at a rate of 3900 rounds per minute (in 2-3 second bursts) and a velocity of 3500 ft/s (1067 m/s), is this true? Assume that "noticeable" means an average recoil force greater than the thrust the Warthog's two TF-34 engines: 18,130 lbs force ≈ 80,600 N.

Solution:
given: $m = 395g = 0.395kg$, $v_0 = 0$, $v_f = 1067m/s$, $t_0 = 0$, $t_f = 1min = 60s$, $= 3900$

unknown: F = ?

$$F = \frac{\Delta p}{\Delta t}$$

$$F = \frac{\Delta (mv)}{\Delta t}$$

$$F = \frac{0.395kg\left(1067m/s\right)}{60s} \times 3900 \text{ rounds} \approx 27{,}400kg \cdot m/s^2$$

$$F \approx 27{,}400N$$

No; the 27,400 N recoil force is less than the 80,900 thrust force.

Problem 2: If a 100 kg linebacker, running at 7.5 m/s, tackles a 150 kg tackling sled, initially at rest, how fast would the combination be immediately moving after the collision?

Solution: "tackle" is an inelastic collision; is 1-dimensional - difference between horizontal and vertical is not important
given: $m_1 = 100kg$, $v_{1i} = 7.5m/s$, $m_2 = 150kg$, $v_{2i} = 0$
unknown: $v_f = ?$

$$m_1\overrightarrow{v_{1i}} + m_2\overrightarrow{v_{2i}} = (m_1 + m_2)\ \overrightarrow{v_f}$$

$$m_1 v_{1i} + m_2\cancel{v_{2i}} = (m_1 + m_2)\ v_f$$

$$\therefore\ v_f = \frac{m_1 v_{1i}}{m_1 + m_2}$$

$$v_f = \frac{100kg\ (7.5m/s)}{100kg + 150kg}$$

$$v_f = 3.0m/s$$

Problem 3: A 15 g bullet is fired at 280 m/s into a 3.20 kg block sliding in the opposite direction at 0.85 m/s on a level surface. The bullet punches it way through the block and emerges traveling at 110 m/s. What new speed does the bullet push the block?

Solution: this is considered an elastic collision, because the masses remain separate.
given:
$m_1 = 15g = 0.015kg$, $m_2 = 3.20kg$, $v_{1i} = 280m/s$, $v_{1f} = 110m/s$, $v_{2i} = -0.85m/s$

unknown: $v_{2f} = ?$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$m_1 v_{1i} + m_2 v_{2i} - m_1 v_{1f} = m_2 v_{2f}$$

$$m_1\left(v_{1i} - v_{1f}\right) + m_2 v_{2i} = m_2 v_{2f}$$

$$\therefore\ v_{2f} = \frac{m_1\left(v_{1i} - v_{1f}\right) + m_2 v_{2i}}{m_2}$$

$$v_{2f} = \frac{0.015kg\ (280m/s - 110m/s) + 3.20kg\ (-0.85m/s)}{3.20kg}$$

$$v_{2f} \approx -0.053m/s$$

Problem 4: A 10 g bullet is fired into a 2.80 kg pendulum hanging at rest from a long wire. The bullet becomes imbedded in the pendulum bob and pushes the pendulum to swing to a maximum height of 42.2 cm. What was the initial velocity of the bullet?

Solution: this is a two part problem. The bullet imbedding in the pendulum is an inelastic collision obeying conservation of momentum. The swinging pendulum obeys conservation of energy. The final velocity of the collision transfers as the initial velocity of the swinging. Since the question asks for initial velocity of the collision, the calculations will work backwards, begin with swinging.

swinging given:

$m_{total} = 0.010kg + 2.80kg = 2.81kg$

$h_i = 0,\ h_f = 42.2cm = 0.422m,\ v_f = 0$

swinging unknown: $v_i = ?$

$$KE_i + PE_i = KE_f + PE_f$$

$$\frac{1}{2}m_t v_i^2 + m_t g \cancel{h_i} = \frac{1}{2}m_t \cancel{v_f}^2 + m_t g h_f$$

$$\frac{1}{2}\cancel{m_t} v_i^2 = \cancel{m_t} g h_f$$

$$\therefore\ v_i = \sqrt{2gh_f}$$

$$v_i = \sqrt{2\left(9.8m/s^2\right)(0.422m)}$$

$$v_i \approx 2.876m/s$$

collision given: $m_b = 10g = 0.010kg,\ m_p = 2.80kg,\ v_{p\,i} = 0,\ v_{collision\ f} = v_{swinging\ i}$

collision unknown: $v_{b\,i} = ?$

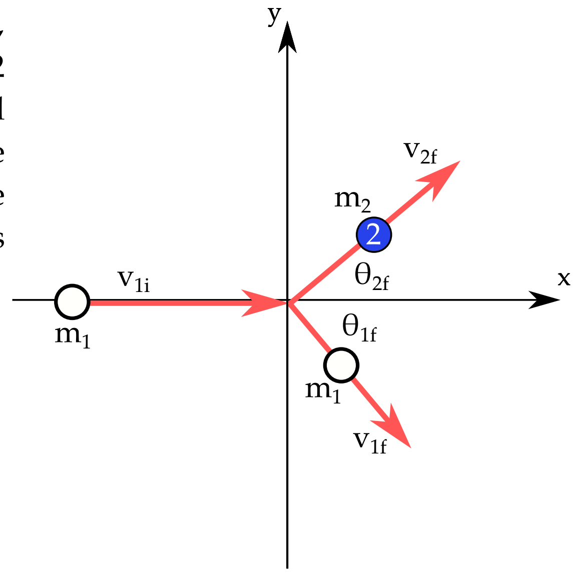
$$m_b v_{b\ i} + m_p \cancel{v_{p\ i}} = \left(m_b + m_p\right) v_f$$

$$v_{b\,i} = \frac{\left(m_b + m_p\right) v_f}{m_b}$$

$$\therefore\ v_{b\,i} = \frac{\left(0.010kg + 2.80kg\right) 2.876m/s}{0.010kg}$$

$$v_{b\,i} \approx 808m/s$$

Problem 5: In a game of pool, the cue ball, traveling at 3.6 m/s collides with the number 2 ball, at rest. It is observed that the 2-ball bounces at 2.7 m/s at 40° away from the cue ball's original path. At what speed and angle does the cue ball bounce? Assume the balls have equal mass.




Solution:

given:
$v_{1i} = 3.6m/s$, $\theta_{1i} = 0^{\circ}$, $v_{2i} = 0$, $m_1 = m_2$,
$v_{2f} = 2.7m/s$, $\theta_{2f} = 40^{\circ}$

unknown: $v_{1f} = ?$, $\theta_{1f} = ?$

Since there are angles, this is a 2-dimensional collision. Momentum must be dealt with as vectors:

$$m_1\overrightarrow{v_{1i}} + m_2\overrightarrow{v_{2i}} = m_1\overrightarrow{v_{1f}} + m_2\overrightarrow{v_{2f}}$$

Beginning with horizontal components:

$$\begin{aligned}
\cancel{m_1} v_{1ix} + \cancel{m_2} v_{2ix} &= \cancel{m_1} v_{1fx} + \cancel{m_2} v_{2fx} \\
v_{1ix} + \cancel{v_{2ix}} &= v_{1fx} + v_{2fx} \\
v_{1i}\cos\theta_{1i} &= v_{1f}\cos\theta_{1f} + v_{2f}\cos\theta_{2f} \\
v_{1i}\cancel{\cos\theta_{1i}} &= v_{1f}\cos\theta_{1f} + v_{2f}\cos\theta_{2f} \\
v_{1i} &= v_{1f}\cos\theta_{1f} + v_{2f}\cos\theta_{2f}
\end{aligned}$$

Since $v_{1f}$ and $\theta_{1f}$ are both unknown, check vertical components to see if the situation simplifies.

$$\begin{aligned}
\cancel{m_1} v_{1iy} + \cancel{m_2} v_{2iy} &= \cancel{m_1} v_{1fy} + \cancel{m_2} v_{2fy} \\
v_{1iy} + \cancel{v_{2iy}} &= v_{1fy} + v_{2fy} \\
v_{1i}\cancel{\sin\theta_{1i}} &= v_{1f}\sin\theta_{1f} + v_{2f}\sin\theta_{2f} \\
0 &= v_{1f}\sin\theta_{1f} + v_{2f}\sin\theta_{2f} \\
\therefore \quad v_{1f} &= \frac{-v_{2f}\sin\theta_{2f}}{\sin\theta_{1f}}
\end{aligned}$$

Transferring back to horizontal components:

$$v_{1i} = \frac{-v_{2f}\sin\theta_{2f}}{\sin\theta_{1f}}\frac{\cos\theta_{1f}}{1} + v_{2f}\cos\theta_{2f}$$
$$v_{1i} = \frac{-v_{2f}\sin\theta_{2f}}{\tan\theta_{1f}} + v_{2f}\cos\theta_{2f}$$
$$3.6m/s = \frac{-2.7m/s\sin 40°}{\tan\theta_{1f}} + 2.7m/s\cos 40°$$
$$3.6m/s = \frac{-1.736m/s}{\tan\theta_{1f}} + 2.068m/s$$
$$1.532m/s = \frac{-1.736m/s}{\tan\theta_{1f}}$$
$$\tan\theta_{1f} = \frac{-1.736m/s}{1.532m/s}$$
$$\tan\theta_{1f} = -1.133$$
$$\theta_{1f} = \tan^{-1}(-1.133)$$
$$\theta_{1f} \approx -48.6°$$

The negative sign indicates the cue ball is moving on the opposite side of the linear axis compared to the 2-ball.

Complete the problem by plugging back to either the horizontal or the vertical equations.

$$v_{1i} = v_{1f}\cos\theta_{1f} + v_{2f}\cos\theta_{2f}$$
$$v_{1f} = \frac{v_{1i} - v_{2f}\cos\theta_{2f}}{\cos\theta_{1f}}$$
$$v_{1f} = \frac{3.6m/s - 2.7m/s\,(\cos 40°)}{\cos\left(-48.6°\right)}$$
$$v_{1f} \approx 2.32m/s$$

$$v_{1f} = \frac{-v_{2f}\sin\theta_{2f}}{\sin\theta_{1f}}$$
$$v_{1f} = \frac{-(2.7m/s)\sin 40°}{\sin\left(-48.6°\right)}$$
$$v_{1f} \approx 2.31m/s$$

Notice there is a small rounding difference between these two answers. Since the algebra is long, there are other paths that can be taken. They will all eventually lead back to one final answer set. In Physics, some Math techniques are considered more “elegant” than others, but we won’t get too worried about that.

## Part 9: Rotation

**"The tone of science [is] magniloquently boastful … creatively confident, generous, argumentative, lavish and full of hope." C. P. Snow (1905 – 1980)**

Up until now, we have been assuming that our moving objects have been moving in a line (linear motion), but were not rotating. Even in centripetal motion, although our object may have traced a circular path, it was not spinning. We are now going to look at spinning objects.

What is most important and what you must understand about this topic is: everything that you learned before is still true! Only certain measurements and therefore certain notations have changed. What you are describing and calculating has not changed! Essentially we are going to re-cast linear motion, Newton's Second Law, work and energy, and momentum in terms of spinning objects. Everything looks different, yet still the same. The only quantity that does not need to be altered is time. Time is still measured in seconds.

Rotational kinetics deals with four major measurements:

angular displacement (angle theta - $\theta$) - how far does the object rotate,
time (t) - how long does it take something to rotate,
angular velocity (omega - $\omega$) - how fast does the object rotate,
angular acceleration (alpha - $\alpha$) - how does angular velocity increase or decrease as the object rotates.

Keep in mind: in more advanced Math topics, angles are not measured in degrees. They are measured in radians. In earlier Physics topics, degrees were OK, but rotational uses radians:

$$\theta = \frac{s}{r}$$

$$2\pi \text{ radians} = 1 \text{ revolution} = 360°$$

Angular (spinning) motion: "Everything is the same as before; it just looks different."

$$\vec{\omega} = \frac{\Delta \vec{\theta}}{\Delta t} \qquad \text{OR} \qquad \vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\theta}}{\Delta t} = \frac{d\vec{\theta}}{dt}$$

$$\vec{\alpha} = \frac{\Delta \vec{\omega}}{\Delta t} \qquad \text{OR} \qquad \vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

And the motion equations are:

$$① \quad \omega_f = \omega_0 + \alpha t$$

$$② \quad \theta_f = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$

$$③ \quad \omega_f^2 = \omega_0^2 + 2\alpha\left(\theta_f - \theta_0\right)$$

Compare these definitions and formulae to those introduced before. Can you see that they are not new? Pay attention to positive versus negative rotation direction. We will continue to follow the conventional angle direction - counterclockwise rotation is positive and clockwise rotation is negative.

ROLLING combines linear and rotational motion. A wheel does not spin in place - as a wheel spins, its axis moves linearly. The radius connects linear and rotational motion.

$$\theta = \frac{s}{r}$$

$$\omega = \frac{v}{r}$$

$$\alpha = \frac{a}{r}$$

The shape of an object affects the way it will spin. The moment of inertia  replaces mass, but depends on the mass and shape.

$$I = \int r^2 \, dm \qquad \text{(PHY 1300 only)}$$

OR $\quad I = kmr^2$ where k is the shape constant

for some simple shapes:

$$I_{disk} = \tfrac{1}{2}mr^2$$

$$I_{solid\ sphere} = \tfrac{2}{5}mr^2$$

$$I_{hollow\ sphere} = mr^2$$

$$I_{hoop} = mr^2$$

For example: a solid wheel is a disk, a baseball is a sold sphere, a basketball is a hollow sphere, a bicycle wheel is a hoop.

center of mass of simple multi-mass objects (PHY 1300 only)

$$x_{cm} = \frac{\sum x_i m_i}{\sum m_i}$$

We continue to replace linear measurements with angular measurements. We substitute TORQUE (tau - τ)for force:

$$\tau = r_{\perp}F = rF\sin\phi_{rF}$$

Strictly speaking, torque is computed as a cross product (PHY 1300 only)

$$\vec{\tau} = \vec{r} \times \vec{F}$$

This means torque exists in a 3-dimensional space. It is normally not necessary to worry about (yet). The magnitude is normally enough.

$$\left|\vec{\tau}\right| = \left|\vec{r} \times \vec{F}\right| = |r| \left|F\right| \sin\phi_{rF}$$

Just pay attention to a positive counterclockwise versus a negative clockwise rotation direction.

Since torque replaces force, we have a new version of Newton's Second Law

$$\sum \vec{\tau} = I\vec{\alpha}$$

Since rotation is a different type of motion, we also have rotational kinetic energy

$$KE = \tfrac{1}{2}I\omega^2$$

The work-energy theorem and conservation of energy is still valid.

Angular momentum is:

$$L = I\omega$$

Conservation of momentum is still valid.

Since "Everything is the same as before; it just looks different" recognize that rotation problems are solved in the exact same way as before. Learn to be comfortable with the new notations.

# Part 9 Rotation Problems

Problem 1: A Ford Mustang GT pony car can accelerate from rest to 100 mph (44.7 m/s) in 9.5 s. (a) What is the angular acceleration of its 70 cm diameter wheels? (b) How many revolutions do the wheels make in this time? (c) How fast are the wheels spinning at the end of the 9.5 s?

Solution:

given: $v_0 = 0$, $\omega_0 = 0$, $v_f = 44.7m/s$, $t = 9.5s$, $r = \frac{d}{2} = 0.35m$

unknown: $\alpha = ?$, $\theta_f$ in rev $= ?$, $\omega_f = ?$

part (a):

$$v_f = \cancel{v_0} + at$$
$$\therefore \quad a = \frac{v_f}{t}$$
$$a = \frac{44.7m/s}{9.5s}$$
$$a \approx 4.71m/s^2$$

$$\alpha = \frac{a}{r}$$
$$\alpha = \frac{4.71m/s^2}{0.35m}$$
$$\alpha \approx 13.4rad/s^2$$

part (b):

$$\theta_f = \cancel{\theta_0} + \cancel{\omega_0}\, t + \tfrac{1}{2}\alpha t^2$$
$$\theta_f = \tfrac{1}{2}\left(13.4rad/s^2\right)\left(9.5s\right)^2$$
$$\theta_f \approx 605rad$$

$$605rad \times \frac{rev}{2\pi rad} \approx 96rev$$

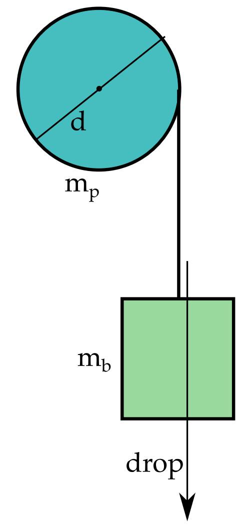
part (c):

$$\omega_f = \frac{v_f}{r}$$
$$\omega_f = \frac{44.7m/s}{0.35m}$$
$$\omega_f \approx 127rad/s$$

OR

$$\omega_f = \cancel{\omega_0} + \alpha t$$
$$\omega_f = 13.4rad/s^2\left(9.5s\right)$$
$$\omega_f \approx 127rad/s$$

Problem 2: A 2 kg block is attached below a 6 kg, 50 cm diameter solid pulley. When the block is released from rest, it unwinds string off the pulley. What length of string will be pulled off the pulley after the block drops for 2 s?



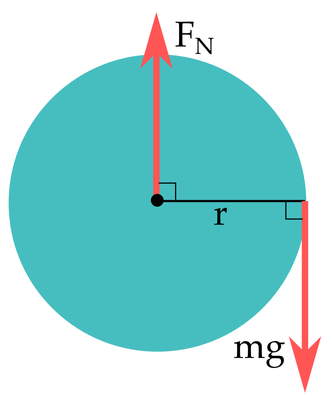

Solution:

given:

$$m_b = 2kg,\ m_p = 6kg,\ r = \frac{d}{2} = 0.25m,\ v_0 = 0,\ \omega_0 = 0,\ t = 2s$$

unknown: $s_f = ?$

Free-body diagram:




$$\begin{aligned} mg &= 19.6N \\ F_N &= 19.6N \\ \phi_{mg} &= 90° \\ \phi_{F_N} &= 90° \\ r_{mg} &= 0.25m \\ r_{F_N} &= 0 \end{aligned}$$

rotation measurements:

$$\begin{aligned} I_{disk} &= \frac{1}{2}mr^2 \\ I_{pulley} &= \frac{1}{2}\left(6kg\right)\left(0.25m\right)^2 \\ I_{pulley} &= 0.1875kg \cdot m^2 \end{aligned}$$

$$\begin{aligned} \tau &= r_\perp F = rF\sin\phi_{rF} \\ \tau_{mg} &= r_{mg}\ mg\sin\phi_{rF} \\ &= 0.25m\left(19.6N\right)\sin 90° \\ \tau_{mg} &= 4.9m \cdot N \\ \tau_{F_N} &= \cancel{r_{F_N}}\ F_N\sin\phi_{rF} \\ \tau_{F_N} &= 0 \end{aligned}$$

Newton's Second Law

$$\sum \tau = I\alpha$$
$$\tau_{F_N} = I\alpha$$
$$\therefore \ \alpha = \frac{\tau_{F_N}}{I}$$
$$\alpha = \frac{4.9m \cdot N}{0.1875kg \cdot m^2}$$
$$\alpha \approx 26.1 rad/s^2$$

motion formula:

$$\theta_f = \cancel{\theta_0} + \cancel{\omega_0} t + \tfrac{1}{2}\alpha t^2$$
$$\theta_f = \tfrac{1}{2}\left(26.1 rad/s^2\right)(2s)^2$$
$$\theta_f \approx 52.2 rad$$

$$\theta_f = \frac{s_f}{r}$$
$$\therefore \ s_f = r\theta_f$$
$$s_f = 0.25m\left(52.2 rad\right)$$
$$s_f \approx 13m$$

Problem 3: A 4.5 kg, 30 cm diameter bowling ball rolls, without slipping, down a 5 m high inclined plane. If the ball rolls from rest at the top, what will be its linear speed at the bottom of the incline?

Solution: if you've never bowled before, a bowling ball is a solid sphere.
given:
$m = 4.5kg,\ r = \dfrac{d}{2} = 0.15m,\ v_0 = 0,\ \omega_0 = 0,\ h_i = 5m,\ h_f = 0$
unknown: $v_f = ?$

There is more than one way to solve this problem. Mathematically, the simplest is considered the use of conservation of energy.

$$\begin{aligned} \cancel{KE_i} + PE_i &= KE_f + \cancel{PE_f} \\ PE_i &= KE_{f\ linear} + KE_{f\ angular} \\ mgh_i &= \tfrac{1}{2}mv_f^2 + \tfrac{1}{2}I\omega_f^2 \end{aligned}$$

$$\text{since} \quad I_{solid\ sphere} = \tfrac{2}{5}mr^2 \quad \text{and} \quad \omega = \frac{v}{r}$$

$$\begin{aligned} \therefore\ \cancel{m}gh_i &= \frac{1}{2}\cancel{m}v_f^2 + \frac{1}{2}\left(\frac{2}{5}\cancel{m}r^2\right)\left(\frac{v_f}{r}\right)^2 \\ gh_i &= \frac{1}{2}v_f^2 + \frac{1}{5}v_f^2 \\ gh_i &= \frac{7}{10}v_f^2 \\ \therefore\ v_f &= \sqrt{\tfrac{10}{7}gh_i} \\ v_f &= \sqrt{\tfrac{10}{7}\left(9.8m/s^2\right)\left(5m\right)} \\ v_f &\approx 8.37m/s \end{aligned}$$

Notice that the given mass is not needed to solve this problem. Scientists and engineers always like to gather as much data as possible, just in case. That means some of the data may turn out to be unnecessary. You don't know whether something is important or not ,until you finish.

# Part 10: Static equilibrium

**“If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.” John von Neumann (1903 – 1957)**

Static equilibrium says “Nothing moves because everything balances.”

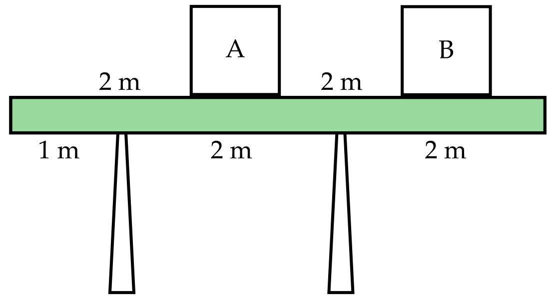
“Nothing moves” means no horizontal, no vertical and no rotational motion. “Everything balances” means all horizontal and vertical forces, plus torques must cancel out. Thus, we have “the three conditions of static equilibrium:”

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum \tau = 0$$

Since these are still about forces, we still use free-body diagrams. We are building on earlier knowledge.

## Part 10 Static equilbrium Problems

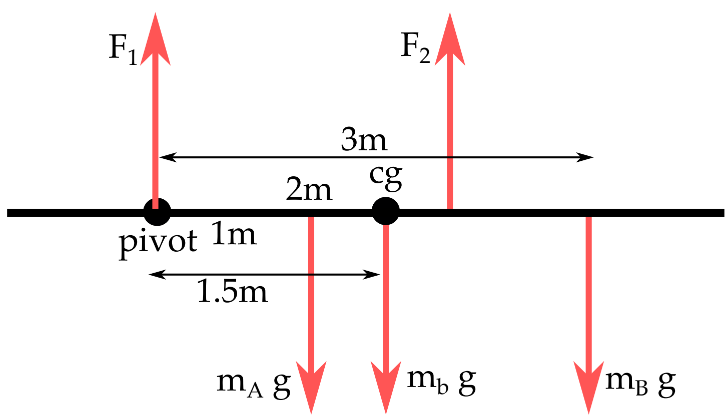
Problem 1: Two 10 kg masses are placed on a 20 kg, 5 m uniform horizontal beam. The beam has two supports. See the figure. What are the two support forces?




Solution:
given: $m_b = 20kg$, $m_A = 10kg$, $m_B = 10kg$

unknown: $F_1 = ?$, $F_2 = ?$

Free-body diagram (find the forces): since there is a possibility of rotation, it is inappropriate to use an x-y plane. You should use a line to represent the beam and place the forces along the line where they belong. Theoretically, you can put the pivot point anywhere you want. To keep things simple, I always choose the pivot at the first support from the left, and measure all radii from there. The center of gravity is the geometric center of a uniform or homogenous object. Objects are uniform in this class, unless specifically stated otherwise.




The diagram's geometry gives: $r_1 = 0$, $r_A = 1m$, $r_b = 1.5m$, $r_2 = 2m$, $r_B = 3m$

Find the torques: since all angles are 90°, all sin ϕ = 1

$$\tau = rF\cancel{\sin\phi_{rF}}$$
$$\tau_{F_1} = \cancel{r_1}\ F_1 = 0$$
$$\tau_{m_A g} = r_A\ m_A g\ = 1m\left(10kg\right)\left(9.8m/s^2\right) = 98m\cdot N\ (CW)$$
$$\tau_{m_b g} = r_b\ m_b g\ = 1.5m\left(20kg\right)\left(9.8m/s^2\right) = 294m\cdot N\ (CW)$$
$$\tau_{F_2} = r_2\ F_2 = 2m\left(F_2\right) = 2F_2\ m\cdot N\ (CCW)$$
$$\tau_{m_B g} = r_B\ m_B g\ = 3m\left(10kg\right)\left(9.8m/s^2\right) = 294m\cdot N\ (CW)$$

Now that the Physics is done; we're ready for the Math:

First, balance the torques:

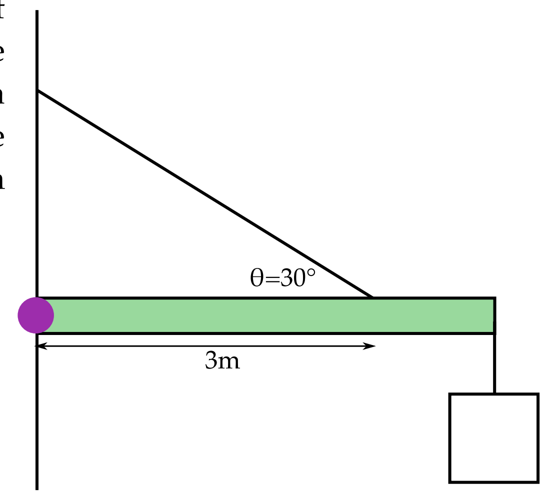
$$\sum \tau \ = \ 0$$
$$\tau_{F_2} - \tau_{m_A} - \tau_{m_b} - \tau_{m_B} = 0$$
$$\tau_{F_2} = \tau_{m_A} + \tau_{m_b} + \tau_{m_B}$$
$$2F_2 = 98 + 294 + 294$$
$$2F_2 = 686$$
$$F_2 = 343m\cdot N$$

Second, balance the vertical forces:

$$\sum F_y \ = \ 0$$
$$F_1 + F_2 - m_A g - m_b g - m_B g = 0$$
$$F_1 = m_A g + m_b g + m_B g - F_2$$
$$F_1 = g\left(m_A + m_b + m_B\right) - F_2$$
$$F_1 = 9.8m/s^2\left(10kg + 20kg + 10kg\right) - 343N$$
$$F_1 = 392N - 343N$$
$$F_1 = 49N$$

Last, balance the horizontal forces - not necessary in this problem.

Problem 2: A 15 kg mass hangs from the end of a 10 kg, 4 m uniform horizontal beam. The beam is attached to a wall by a cable and a hinge. See the figure. What is the tension in the cable, and the horizontal and vertical forces on the hinge?

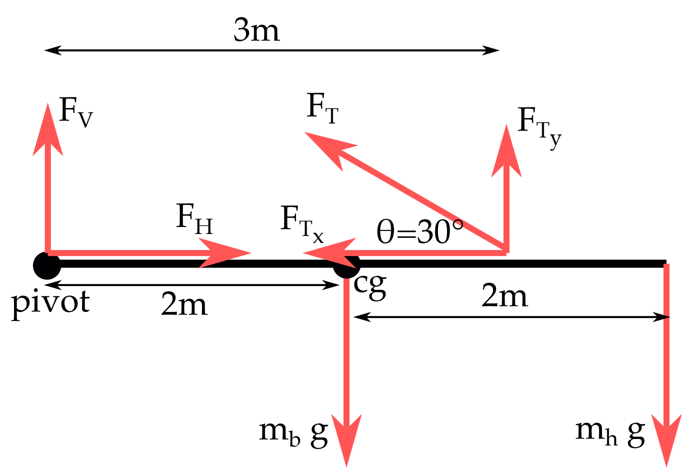



Solution:

given: $m_b = 10kg$, $m_h = 15kg$, $\theta_T = 30°$

unknown: $F_T = ?$, $F_H = ?$, $F_V = ?$

Free-body diagram (find the forces): the line represents the beam. The hinge is the natural place to put the pivot point.




The diagram's geometry gives: $r_V = 0$, $r_H = 0$, $r_b = 2m$, $r_T = 3m$, $r_h = 4m$, $F_H$ and $F_{Tx}$ are parallel to the beam - their angles $\phi$ are 0°, and their sin $\phi = 0$, and the other angles are perpendicular, with sin $\phi = 1$.

Find the torques:

$$\tau = rF \sin\phi_{rF}$$

$$\tau_{F_V} = \cancel{r_V}\ F_V\ \sin\phi_V = 0$$

$$\tau_{F_H} = \cancel{r_H}\ F_H\ \cancel{\sin\phi_H} = 0$$

$$\tau_{m_b g} = r_b\ m_b g \sin\phi_b = 2m\left(10kg\right)\left(9.8m/s^2\right) = 192m\cdot N\ (CW)$$

$$\tau_{F_{T_y}} = r_T\ F_T \sin\theta_T\ \ \sin\phi_{T_y} = 3m\left(F_T\right)\sin 30° = 1.5F_T\ m\cdot N\ (CCW)$$

$$\tau_{F_{T_x}} = r_T\ F_T \cos\theta_T\ \ \cancel{\sin\phi_{T_x}} = 0$$

$$\tau_{m_h g} = r_h\ m_h g \sin\phi_h = 4m\left(15kg\right)\left(9.8m/s^2\right) \approx 588m\cdot N\ (CW)$$

Notice, although the free-body diagram is complex, there are only three torques, because the rest compute to zero.

Finish the Physics; do the Math:

First, balance the torques:

$$\sum \tau = 0$$

$$\tau_{T_y} - \tau_{m_b} - \tau_{m_h} = 0$$

$$\tau_{T_y} = \tau_{m_b} + \tau_{m_h}$$

$$1.5F_T = 192 + 588$$

$$1.5F_2 = 780$$

$$F_T = 520m\cdot N$$

Second, balance the vertical forces:

$$\sum F_y = 0$$

$$F_V + F_{T_y} - m_b g - m_h g = 0$$

$$F_V = m_b g + m_h g - F_{T_y}$$

$$F_V = g\left(m_b + m_h\right) - F_T \sin\theta_T$$

$$F_V = 9.8m/s^2\left(10kg + 15kg\right) - 520N\ \sin 30°$$

$$F_V \approx 245N - 260N$$

$$F_V = -15N$$

Notice, this is slightly negative, because the hanging mass on the right side is heavy enough to swing the left side upward, requiring the hinge to pull back downward.

Last, balance the horizontal forces.

$$\begin{aligned}\sum F_x &= 0\\ F_H - F_{T_x} &= 0\\ F_H &= F_{T_x}\\ F_H &= F_T \cos\theta_T\\ F_H &= 520N \cos 30^\circ\\ F_H &= 450N\end{aligned}$$

# Part 11: Fluids

**"It is a scientist's duty to be optimistic." Edwin Land (1909 – 1991)**

A fluid is any liquid or gas. Fluids do not have definite shape - their shape changes easily with applied forces. We say that fluids can flow.

For fluids, mass measurements may not be enough. Fluids are usually measured by their density, which is mass per volume:

$$\rho = \frac{m}{V}$$

Notice, Physicists don't use "d" to abbreviate density – its too easily confused with displacement or derivative. We use "rho" – **ρ** (which looks like a "p").

Density is also discussed in Chemistry, but the units in Chemistry are $g/cm^3$, but $kg/m^3$ in Physics. The conversion factor is 1000 kg/m3 = 1 g/cm3. Chemists do abbreviate "d" for density.

"It is a somewhat dirty secret in science, that there are actually two different metric systems: the one used in Physics and the one used in Chemistry."

Liquids are INCOMPRESSIBLE fluids. Their volume (and density) remains constant under changing pressure (unless the pressure becomes extremely high).

Gases are COMPRESSIBLE fluids. Their volume is inversely proportional to pressure (Boyle's Law from chemistry) and their density increases with increasing pressure. Since the atoms/molecules of a gas are widely spaced (discussed in Chemistry), the density of a gas is about 1000 times less than the density of liquids.

---

## Fluid Statics

Static means not moving – static fluids are "still", or at rest.

Since all matter has mass, fluids also have a weight on the Earth. We say that this weight exerts a pressure on immersed object.

$$P = \frac{F}{A}$$

PASCAL'S PRINCIPLE says that liquid pressure is proportional to depth of a fluid (height above immersed object):

$$P = \rho g h$$

ARCHIMEDES' PRINCIPLE says that a upward buoyant force results from the pressure differential between the top and bottom of any object immersed in a fluid

$$F_{bouyant} = \rho_{fluid}\, g V_{displaced}$$

Buoyancy explains why balloons float in the air and why ships float in water. If the maximum upward buoyant force is greater than or equal to the downward weight, an object will float. Displaced volume is the volume of the object surrounded by the fluid. If an object is not completely immersed, the object's total volume is not equal to the displaced volume. For example, ships have a draft (the height of the ship that is underwater) and a freeboard (the height of the ship that is above the water level).

If the density of an object is higher than the density of the fluid, the upward buoyant force is less than the downward weight, so the object will sink. However, it will feel lighter than normal.

$$mg_{apparent} = mg_{real} - F_{bouyant}$$

---

## Fluid Dynamics

Dynamic means moving – dynamic fluids are "flowing", they are not at rest.

BERNOULLI'S PRINCIPLE relates the pressure and velocity changes of dynamic fluids, assuming LAMINAR FLOW - the fluid moves smoothly and is not TURBULENT.

$$P_1 + \rho g h_1 + \tfrac{1}{2}\rho v_1^2 = \rho g h_2 + \tfrac{1}{2}\rho v_2^2 + P_2$$

Be aware, real fluid flow is usually turbulent, and what we do in this class is a very rough approximation.

Bernoulli's Principle is very useful, because it helps explain how an airplane flies. Assuming laminar (smooth) flow, if an airplane wing is cambered (curved) on the top and flat on the bottom, the velocity of the air flow over the top of the wing is much faster than under the wing bottom. The velocity difference produces a pressure difference, resulting in a net upward lift force. (Note, this a partial explanation. There are other things happening, which we do not discuss.)

## Part 11 Fluid Problems:

Problem 1: What is the weight of the air inside a living room of 4 m × 6 m × 3 m dimensions. The density of air is 1.20 kg/m3.

Solution: given: $V = 4m \times 6m \times 3m$, $\rho = 1.20kg/m^3$

unknown: $weight = mg = ?$

$$\rho = \frac{m}{V}$$
$$m = \rho V$$
$$\therefore \; mg = \rho g V$$
$$mg = 1.20kg/m^3 \left(9.8m/s^2\right)\left(4m \times 6m \times 3m\right)$$
$$mg = 847N \approx 190lb$$

Problem 2: Hoover Dam, on the Colorado River, controls Lake Mead, the largest reservoir in America (37 billion cubic meters of water), for irrigation and hydroelectric power for southern California, Nevada and Arizona. What would be the total force pushing water through a 2 m diameter underwater tunnel through the dam, 200 m below the water level. The density of fresh water is $1000kg/m^3$.

Solution: given: $r = \dfrac{d}{2} = 1m$, $h = 200m$, $\rho = 1000kg/m^3$

unknown: $F = ?$

Since $P = \dfrac{F}{A}$ and $P = \rho g h$

$$\frac{F}{A} = \rho g h$$
$$\therefore \; F = \rho g h A$$

then

$$F = \rho g h \; \pi r^2$$
$$F = 1000kg/m^3 \left(9.8m/s^2\right)(200m)(3.14)(1m)^2$$
$$F \approx 6.15 \times 10^6 N$$

Problem 3: A persistent complaint about the 1997 movie Titanic is near the end, when Jack saves Rose by pushing her on a door from the wreckage to use as a raft, but does not save himself by also getting on the door. Could Jack have climbed aboard the door without causing it to sink and saved himself? Assume that Rose and Jack have masses of 65 kg and 85 kg, respectively, and it is a solid oak wood door, 3 m x 2 m x 10 cm. Oak has a density of 750 kg/m³; seawater density is 1025 kg/m³.

Solution: given:

$$\begin{aligned} m_{Rose} &= 65kg,\ m_{Jack} = 85kg, \\ V_{displaced} &= V_{seawater} = V_{oak} = 3m \times 2m \times 0.10m \\ \rho_{oak} &= 750kg/m^3,\ \rho_{seawater} = 1025kg/m^3 \end{aligned}$$

unknown: $F = ?$

$$\begin{aligned} F_{bouyant} &= (mg)_{door} + (mg)_{Rose} + (mg)_{Jack} \\ (\rho g V)_{seawater} &= (\rho g V)_{oak} + (mg)_{Rose} + (mg)_{Jack} \\ \rho V_{seawater} &= \rho V_{oak} \ + \ m_{Rose} \ + \ m_{Jack} \\ (1025kg/m^3)(3m)(2m)(0.10m) &= (750kg/m^3)(3m)(2m)(0.10m) + 65kg + 85kg \\ 615kg &= 450kg \ + \ 65kg \ + \ 85kg \\ 615kg &> 600kg \end{aligned}$$

YES, the maximum mass that could be supported by the buoyant force is greater than the total mass. (However, the safety margin is dangerously small.)

Problem 4: The largest airships ever built in the United States were USS's Akron and Macon of the 1930's US Navy. They were 785 ft long and held 6,500,000 cubic feet (184,000 cubic meters) of helium gas. Akron had a mass of 110,000 kg. What was her maximum useful load (crew, fuel, supplies)? The density of air and helium are 1.20 kg/m³ and 0.166 kg/m³, respectively.


Solution:

given:
$m_{airship} = 110{,}000 kg$, $V_{displaced} = V_{air} = V_{helium} = 184{,}000 m^3$,
$\rho_{air} = 1.20 kg/m^3$, $\rho_{helium} = 0.166 kg/m^3$

unknown: $m_{load} = ?$

$$
\begin{aligned}
F_{bouyant} &= (mg)_{airship} + (mg)_{helium} + (mg)_{load} \\
(\rho g V)_{air} &= (m\cancel{g})_{airship} + (\rho \cancel{g} V)_{helium} + (m\cancel{g})_{load} \\
\rho V_{air} &= m_{airship} + \rho V_{helium} + m_{load} \\
\therefore\ m_{load} &= \rho V_{air} - m_{airship} - \rho V_{helium} \\
&= 1.20\ kg/m^3 \left(184{,}000\ m^3\right) \\
&\quad -110{,}000\ kg - 0.166\ kg/m^3 \left(184{,}000\ m^3\right) \\
&\approx 220{,}800\ kg - 110{,}000\ kg - 30{,}540\ kg \\
m_{load} &\approx 80{,}300\ kg
\end{aligned}
$$

Problem 5: The largest twin-engine airplane ever made is the Boeing 777. It can carry 375 passengers over 8400 miles at 550 miles per hour (246 m/s). If half the total lift force on a 683,000 lb (310,000 kg) Triple-Seven cruising at 36,000 ft (11,000 m) altitude derives from Bernoulli's Principle, what would be the air velocity over the top of the wings with an area of 4700 ft2 (437 m2). Assume the wing thickness is negligible. The air density at 11,000 m is about 25% the 1.20 kg/m3 sea level value.

Solution:

given: $v_1 = 246\ m/s,\ m = 310{,}000\ kg,\ A = 437\ m^2,$

$$\rho = 1.20\ \frac{kg}{m^3} \times 0.25 = 0.30\ \frac{kg}{m^3},\ h_1 \approx h_2$$

unknown: $v_2 = ?$

$$\begin{aligned} P_1 + \cancel{\rho g h_1} + \tfrac{1}{2}\rho v_1^2 &= \cancel{\rho g h_2} + \tfrac{1}{2}\rho v_2^2 + P_2 \\ P_1 - P_2 &= \tfrac{1}{2}\rho v_2^2 - \tfrac{1}{2}\rho v_1^2 \\ -\Delta P \equiv \frac{-\Delta F}{A} \equiv \frac{\frac{1}{2}mg}{A} &= \tfrac{1}{2}\rho v_2^2 - \tfrac{1}{2}\rho v_1^2 \\ \therefore \quad v_2 &= \sqrt{\frac{mg}{A\rho} + v_1^2} \end{aligned}$$

$$\begin{aligned} v_2 &= \sqrt{\frac{(310{,}000\ kg)\,(9.8\ m/s^2)}{(437\ m^2)\,(0.30\ kg/m^3)} + (246\ m/s)^2} \\ &\approx \sqrt{23173\ m^2/s^2 + 60{,}516\ m^2/s^2} \\ v_2 &\approx 289\ m/s \end{aligned}$$

# Part 12: Simple harmonic motion

**"The scientific method allows ordinary people to do extraordinary things." Francis Bacon (1561 – 1626)**

In this topic, "motion" still means moving, "harmonic" means the motion repeats back-and-forth, and "simple" means the motion only repeats, forever (theoretically). Simple harmonic motion is abbreviated SHM, and is also called PERIODIC MOTION.

There are many common occurrences of harmonic motion: a guitar string when plucked, a tree swaying in the wind, the ground shaking during an earthquake, a playground swing, etc.; although real-life examples may not be simple.

Simple harmonic motion is described:

> the EQUILIBRIUM position is the natural position of the object, if it were at rest.
>
> x - the DISPLACEMENT is the position change to either side of the equilibrium. Strictly speaking, one side is positive and the other negative. However, simple harmonic motion is symmetric, and the sign values are usually ignored. Note, the displacement is always written as x, whether it is horizontal or vertical.
>
> A - the AMPLITUDE is the maximum displacement to either side from of the equilibrium. Since we assume SHM is symmetric, the Amplitude is the same on both sides of the equilibrium.
>
> v - the VELOCITY is change of position (over time). SHM velocity is not constant - the maximum velocity occurs as the motion zips through equilibrium, the minimum (zero) velocity occurs as the motion turns around at the Amplitude.
>
> a - the ACCELERATION is change of velocity (over time). SHM acceleration is not constant - the maximum acceleration occurs as the motion turns around at the Amplitude, the minimum (zero) acceleration occurs as the motion passes through equilibrium.
>
> an OSCILLATION is one complete back-and-forth motion, from any initial position and then back-and-forth to the original position.
>
> T - the PERIOD is the time (in seconds) required for one oscillation.
>
> f - the FREQUENCY is the number of oscillations per second (in cycles per second - $s^{-1}$, or Hertz - Hz).

Since rigid objects can't bend, we usually model SHM using a spring-mass oscillator (abbreviated SMO). We imagine a mass attached to a spring. If the mass is pulled from equilibrium, work is done to stretch or compress the spring, giving it elastic potential energy. Therefore, when the mass is released, it will oscillate, because the elastic spring provides exerts an elastic force and transfers its elastic potential energy into the mass' kinetic energy.

We assume that a spring-mass oscillator:

obeys conservation of energy.
the spring provides elastic potential energy.
the moving mass has kinetic energy

Note, we usually ignore gravitational potential energy in a vertical SMO.

A pendulum is also a good oscillator. We imagine a mass hanging on a spring. If the mass is moved from equilibrium (vertical), work is done to move the mass upward, transferring gravitational potential energy. Therefore, when the mass is released, it will oscillate, because the gravitational force changes the mass' gravitational potential energy into kinetic energy.

We assume that a pendulum:

obeys conservation of energy.
the pendulum height provides gravitational potential energy.
the moving mass has kinetic energy

The equation of simple harmonic motion (or displacement function) always looks like this:

$$x(t) \ = \ A\cos\left(2\pi f t\right) \qquad \text{OR} \qquad x(t) \ = \ A\sin\left(2\pi f t\right)$$

where x is the displacement from equilibrium,
A is the Amplitude
f is the frequency,
t is the time for a particular position x,
and the angular frequency, omega, $\omega = 2\pi f$

For a spring-mass oscillator:

Hooke's Law tells us:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

where k is the spring constant and m is the mass.

Since $f = \frac{1}{T}$ , we can also write:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Conservation of energy gives us:

$$v_{max} = A\sqrt{\frac{k}{m}} \quad \text{and} \quad v = \sqrt{\frac{k}{m}\left(A^2 - x^2\right)} = v_{max}\sqrt{1 - \frac{x^2}{A^2}}$$

the definition of velocity gives us (PHY 1300 only):

$$v(t) = \frac{d\ x(t)}{dt} = 2\pi f\ A \sin\left(2\pi f t\right) \quad \text{and} \quad v_{max} = 2\pi f A$$

And Newton's Second Law says

$$a_{max} = A\frac{k}{m} \quad \text{and} \quad a = x\frac{k}{m}$$

the definition of acceleration gives us (PHY 1300 only):

$$a(t) = \frac{d\ v(t)}{dt} = -4\pi^2 f^2\ A\cos\left(2\pi f t\right) \quad \text{and} \quad a_{max} = 4\pi^2 f^2\ A$$

Note, all these formulae are stated without proof.

For a pendulum, with a small Amplitude (A « L), we have:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \quad \text{OR} \quad T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the pendulum cord length.

# Part 12 Simple Harmonic Motion Problems

Problem 1: A 300. g stone is suspended from a vertical spring and set in motion. Measurements show the maximum speed of the vibrating stone is 35.0 cm/s and the period is 0.400 s. What are the (a) spring constant of the spring, (b) amplitude of the motion, and (c) frequency of oscillation.

Solution:
given: $m = 0.300kg$, $v_{max} = 0.350m/s$, $T = 0.400s$
unknown: $k = ?$, $A = ?$, $f = ?$

part (a):

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad k = \frac{4\pi^2 m}{T^2}$$

$$T^2 = 4\pi^2\,\frac{m}{k} \qquad k = \frac{4\pi^2\left(0.200kg\right)}{(0.400s)^2}$$

$$\therefore\ k = \frac{4\pi^2 m}{T^2} \qquad k \approx 74.0N/m$$

part (b):

$$v_{max} = A\sqrt{\frac{k}{m}}$$

$$v_{max}^2 = A^2\frac{k}{m} \qquad A = \sqrt{\frac{mv_{max}^2}{k}}$$

$$\frac{mv_{max}^2}{k} = A^2 \qquad A = \sqrt{\frac{0.300kg\left(0.350m/s\right)^2}{74.0N/m}}$$

$$A \approx 0.0223m$$

$$\therefore\ A = \sqrt{\frac{mv_{max}^2}{k}}$$

part (c):

$$f = \frac{1}{T}$$

$$f = \frac{1}{0.400s}$$

$$f = 2.50Hz$$

Problem 2: A loudspeaker diagram is observed to be vibrating in simple harmonic motion with a total displacement of 0.50 mm while playing note middle C, 262 Hz. What are the (a) angular frequency, (b) maximum speed, and (c) magnitude of the maximum acceleration?

Solution:

given: $A = \frac{d}{2} = 0.25mm = 1.25 \times 10^{-4}m,\ f = 262Hz$

unknown: $\omega = ?,\ v_{max} = ?,\ a_{max} = ?$

part (a):

$$\omega = 2\pi f$$
$$\omega = 2\pi \left(262Hz\right)$$
$$\omega \approx 1650rad/s$$

part (b):

$$v_{max} = 2\pi f A$$
$$v_{max} = 2\pi \left(262Hz\right)\left(1.25 \times 10^{-4}m\right)$$
$$v_{max} \approx 0.206m/s$$

part (c):

$$a_{max} = 4\pi^2 f^2\ A$$
$$a_{max} = 4\pi^2 \left(262Hz\right)^2 \left(1.25 \times 10^{-4}m\right)$$
$$a_{max} \approx 339m/s^2$$

# Part 13: Heat and temperature

**"Truth is stranger than fiction, but it is because Fiction is obliged to stick to possibilities; Truth isn't." Mark Twain [né Samuel Langhorne Clemens] (1835-1910)**

There is a difference between how temperature is defined depending on whether we are talking about the macroscopic world ("large" things) versus the microscopic world ("small" things, usually atomic/molecular scale).

Macroscopic temperature measures heat stored in a collective object. Heat is a form of energy that is usually considered "inaccessible", because it has a natural "flow" (from hot to cold, by radiation, conduction or convection) that is difficult to interrupt and use to do work.

A large part of engineering is to develop machines to redirect the natural transfer of heat and do useful work - the term is "heat engine."

Microscopic temperature measures the kinetic energy atoms and molecules. On a microscopic scale, temperature is proportional to kinetic energy. We will not being worrying about this.

We are concerned with how heat affect the physical properties of matter.

CALORIMETRY:

A gain or loss of heat can change temperature, depending on a substance's specific heat - s.

$$Q = m\ s\ \Delta T$$

A gain or loss of heat can change phase (solid/liquid/gas), depending on a substance's heat of vaporization or fusion - H.

$$Q = m\ H$$

Since Chemistry is the study of different substances, calorimetry is also discussed there.

THERMAL EXPANSION:

A gain or loss of heat can change the length of an object, depending on a substance's linear expansion constant - $\alpha$.

$$\Delta L = \alpha L_0 \Delta T$$

A gain or loss of heat can change the volume of an object depending on a substance's volume expansion constant - β.

$$\Delta V = \beta V_0 \Delta T$$

## Part 13 Heat Problems

Problem 1: What mass of water ice originally at -20.°C can be heated to hot liquid water at 90.°C by the burning of one gallon (3.785 L = 0.003785 m$^3$) gasoline? The density of gasoline is 740 kg/m$^3$ and the energy of combustion of gasoline (isooctane) is 47,800 kJ/kg. The heat of fusion of $H_2O$ is 333 kJ/kg, the specific heat of solid $H_2O$ is 2.10 kJ/kg · °C and the specific heat of liquid $H_2O$ is 4.18 kJ/kg · °C.

Solution: given:

$V_{gas} = 0.003785\ m^3,\ \rho_{gas} = 740\ kg/m^3,\ Q_{gas} = 47{,}800\ kJ/kg,$

$s_{ice} = 2.10\ kJ/kg \cdot {}^\circ C,\ T_{i\ ice} = -20.{}^\circ C,\ s_{water} = 4.18\ kJ/kg \cdot {}^\circ C,\ T_{f\ water} = 90.{}^\circ C,$

$H_{fusion} = 333\ kJ/kg,\ T_{melt} = 0^\circ C$

unknown: m = ?

$$Q_{gas} = \frac{47{,}800\ kJ}{\cancel{kg}} \times \frac{740\ \cancel{kg}}{\cancel{m^3}} \times 0.003785\ \cancel{m^3} \approx 1.339 \times 10^5\ kJ$$

$$\begin{aligned} Q_{gas} &= Q_{ice} + Q_{melt} + Q_{water} \\ &= (m\ s\ \Delta T)_{ice} + (m\ H)_{melt} + (m\ s\ \Delta T)_{water} \end{aligned}$$

$$\therefore\ m = \frac{Q_{gas}}{(s\ \Delta T)_{ice} + H_{fusion} + (s\ \Delta T)_{water}}$$

$$m \approx \frac{1.339 \times 10^5\ kJ}{\left(2.10\ \frac{kJ}{kg \bullet {}^\circ C}\right)(20.{}^\circ C) + 333\ \frac{kJ}{kg} + \left(4.18\ \frac{kJ}{kg \bullet {}^\circ C}\right)(90.{}^\circ C)}$$

$$m \approx \frac{1.339 \times 10^5\ kJ}{751\ kJ/kg}$$

$$m \approx 178\ kg$$

Problem 2: In New York, a winter temperature of -5°C is normal. In the summer, 35°C is expected. Structures exposed to the elements must account for these temperature changes. The span between the towers of the Verrazano Bridge is 4260 ft. If the span is a straight continuous steel beam, how long is the span in the summer, if the winter length is 1298.4 m. The coefficient of linear expansion of steel is $1.2 \times 10^{-5}\ {}^{\circ}C^{-1}$1.2E-5 °C-1

given: $\alpha = 1.2 \times 10^{-5}\ {}^{\circ}C^{-1}$, $L_0 = 1298.4m$, $\Delta T = 35^{\circ}C - \left(-5^{\circ}C\right) = 40^{\circ}C$

unknown: $L_f = ?$

$$\Delta L = \alpha L_0 \Delta T$$
$$= 1.2 \times 10^{-5}\ {}^{\circ}C^{-1}(1298.4m)(40^{\circ}C)$$
$$\Delta L \approx 0.62m$$

$$L_f = L_0 + \Delta L$$
$$L_f = 1298.4m + 0.62m$$
$$L_f \approx 1299.0m$$

Note: bridges are protected against thermal expansion buckling by “expansion joints” (the steel is not continuous - small gaps are regularly spaced into the span), and by allowing the bridge to arc slightly higher in the summer. A more serious problem is differential expansion. Real structures are not homogeneous - all made of the same material. Different materials expand at different rates. The composite structure can break itself if the expansion difference becomes too large.

Problem 3: The coefficient of volume expansion of water is $2.14 \times 10^{-4}\ {}^{\circ}C^{-1}$, and steel is $3.6 \times 10^{-5}\ {}^{\circ}C^{-1}$. Since water's coefficient is higher, it expands faster than steel. If exactly 1 quart (0.946 Liters) of cold water at 5°C is placed into an exactly 1 Liter steel pot, also at 5°C, would the water expand enough to overflow the pot, if the water and pot are heated to 100°C?

Solution:

given: $\beta_{water} = 2.14 \times 10^{-4}\ {}^{\circ}C^{-1},\ \beta_{steel} = 3.6 \times 10^{-5}\ {}^{\circ}C^{-1},$
$V_{water} = 0.946L,\ V_{steel} = 1L$

unknown: $is\ V_{f\ water} > V_{f\ steel}?$

$$\begin{aligned}\Delta V &= \beta V_0\, \Delta T \\ &= 2.14 \times 10^{-4}\ {}^{\circ}C^{-1}\left(0.946L\right)\left(95^{\circ}C\right) \\ \Delta V_{water} &\approx 0.019L\end{aligned}$$

$$\begin{aligned}\Delta V &= \beta V_0\, \Delta T \\ &= 3.6 \times 10^{-5}\ {}^{\circ}C^{-1}\left(1L\right)\left(95^{\circ}C\right) \\ \Delta V_{steel} &\approx 3.4 \times 10^{-3}L\end{aligned}$$

$$\begin{aligned}V_f &= V_0 + \Delta V \\ V_f &= 0.946L + 0.019L \\ V_{f\ water} &= 0.965L\end{aligned}$$

$$\begin{aligned}V_f &= V_0 + \Delta V \\ V_f &= 1L + 3.4 \times 10^{-3}L \\ V_{f\ steel} &\approx 1.003L\end{aligned}$$

NO, the water overflow will not the pot, because 0.965 L is less than 1.003 L.

## Formulae

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$A_x = \left|A\right| \cos\theta_A \qquad A_y = \left|A\right| \sin\theta_A$$

$$\left|A\right| = \sqrt{A_x^2 + A_y^2} \qquad \theta_A = \arctan\left|\frac{A_y}{A_x}\right| \equiv \tan^{-1}\left|\frac{A_y}{A_x}\right|$$

$$\vec{A} + \vec{B} = \vec{R}$$

$$A_x + B_x = R_x = \left|A\right| \cos\theta_A + \left|B\right| \cos\theta_B$$

$$A_y + B_y = R_y = \left|A\right| \sin\theta_A + \left|B\right| \sin\theta_B$$

$$\vec{A} \bullet \vec{B} = R = \left|A\right| \left|B\right| \cos\phi_{AB}$$

$$\vec{A} \bullet \vec{B} = R = A_x B_x + A_y B_y + A_z B_z$$

$$\vec{A} \times \vec{B} = \vec{R} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

$$\left|\vec{A} \times \vec{B}\right| = \left|\vec{R}\right| = \left|A\right| \left|B\right| \sin\phi_{AB}$$

$$v = \frac{\Delta x}{\Delta t} \qquad a = \frac{\Delta v}{\Delta t}$$

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

$$a = \frac{dv}{dt} \equiv \frac{d^2x}{dt^2}$$

$$j = \frac{da}{dt} \equiv \frac{d^2v}{dt^2} \equiv \frac{d^3x}{dt^3}$$

$$v_f = v_0 + at$$
$$x_f = x_0 + v_0 t + \tfrac{1}{2}at^2$$
$$v_f^2 = v_0^2 + 2a(x_f - x_0)$$

$$g_{Earth} = 9.8 m/s^2 \text{ (down)}$$
$$x = v_{0x} t$$

$$v_{fy} = v_{0y} - gt$$
$$y_{fy} = y_0 + v_{0y} t - \tfrac{1}{2}gt^2$$
$$v_{fy}^2 = v_{0y}^2 - 2g(y_f - y_0)$$

$$\sum \vec{F} = m\vec{a}$$

$$\sum F_x = ma_x$$
$$\sum F_y = ma_y$$

$$weight = mg$$
$$F_{fr} \le \mu F_N$$

$$a_c = \frac{v^2}{r}$$
$$\sum F_c = m\frac{v^2}{r}$$

$$F_g = G\frac{m_1\, m_2}{r^2}$$
$$g_{planet} = G\frac{m_{planet}}{r_{planet}^2}$$
$$v_{satellite} = \sqrt{G\frac{m_{planet}}{r_{orbit}}}$$

$$T_{satellite} = \frac{2\pi r_{orbit}}{v_{satellite}}$$

$$W = F_{\parallel}d = F\,d\cos\phi_{Fd}$$

$$KE = \tfrac{1}{2}mv^2$$

$$PE_{grav} = mgh$$

$$F_{elastic} = -kx$$

$$PE_{elastic} = \tfrac{1}{2}kx^2$$

$$W_{external} + KE_i + PE_i = KE_f + PE_f + W_{fr}$$

$$KE_i + PE_i = KE_f + PE_f$$

$$P = \frac{W}{t}$$

$$p = mv$$

$$\sum F = \frac{\Delta p}{\Delta t}$$

$$J = F\,\Delta t = \Delta(mv)$$

$$m_1v_{1i} + m_2v_{2i} = (m_1 + m_2)\;v_f$$

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$$

$$\theta = \frac{s}{r}$$

$$\omega = \frac{v}{r}$$

$$\alpha = \frac{a}{r}$$

$$\omega_f = \omega_0 + \alpha t$$

$$\theta_f = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$

$$\omega_f^2 = \omega_0^2 + 2\alpha(\theta_f - \theta_0)$$

$$\begin{aligned} I_{disk} &= \tfrac{1}{2} m r^2 \\ I_{solid\ sphere} &= \tfrac{2}{5} m r^2 \\ I_{hollow\ sphere} &= m r^2 \\ I_{hoop} &= m r^2 \end{aligned}$$

$$\tau = r_{\perp} F = rF \sin \phi_{rF}$$

$$\sum \tau = I\,\alpha$$

$$KE = \tfrac{1}{2} I \omega^2$$

$$L = I \omega$$

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum \tau = 0$$

$$\rho = \frac{m}{V}$$

$$P = \frac{F}{A}$$

$$P = \rho g h$$

$$F_{bouyant} = \rho g V$$

$$m g_{apparent} = m g_{real} - F_{bouyant}$$

$$P_1 + \rho g h_1 + \tfrac{1}{2} \rho v_1^2 = \rho g h_2 + \tfrac{1}{2} \rho v_2^2 + P_2$$

$$x(t) = A \cos\left(2\pi f t\right)$$

$$v(t) = \frac{d\ x(t)}{dt} = 2\pi f\, A \sin\left(2\pi f t\right)$$

$$a(t) = \frac{d\ v(t)}{dt} = -4\pi^2 f^2\ A \cos\left(2\pi f t\right)$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$v_{max} = A\sqrt{\frac{k}{m}}$$

$$v_{max} = 2\pi f A$$

$$v = \sqrt{\frac{k}{m}\left(A^2 - x^2\right)} = v_{max}\sqrt{1 - \frac{x^2}{A^2}}$$

$$a_{max} = A\frac{k}{m}$$

$$a_{max} = 4\pi^2 f^2 \, A$$

$$a = x\frac{k}{m}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$Q = m \; s \; \Delta T$$

$$Q = m \; H$$

$$\Delta L = \alpha L_0 \, \Delta T$$

$$\Delta V = \beta V_0 \, \Delta T$$